Multiple transitivity except for a system of imprimitivity

Colin D. Reid[1]

The University of Newcastle
School of Information and Physical Sciences
Callaghan, NSW 2308, Australia
Email: colin@reidit.net

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Abstract

Let Ω be a set equipped with an equivalence relation $\sim$; we refer to the equivalence classes as blocks of Ω. A permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is k**-by-block-transitive** if $\sim$ is G-invariant, with at least k blocks, and G is transitive on the set of k-tuples of points such that no two entries lie in the same block. The action is **block-faithful** if the action on the set of blocks is faithful.

In this article we classify the finite block-faithful 2-by-block-transitive actions. We also show that for $k \geqslant 3$, there are no finite block-faithful k-by-block-transitive actions with nontrivial blocks.

1 Introduction

Given a group G acting on a set X and $x_1, x_2, \dots, x_n \in X$, write $G(x_1)$ for the stabilizer of x_1 in G and $G(x_1, \dots, x_n) = \bigcap_{i=1}^{n} G(x_i)$. Let Ω be a set equipped with an equivalence relation $\sim$; we refer to the equivalence classes of $\sim$ as **blocks** of Ω. Given $\omega \in \Omega$, write $[\omega]$ for the $\sim$-class of ω. For $k \geqslant 1$, define the set $\Omega^{[k]}$ of **distant** k**-tuples** to consist of those k-tuples $(\omega_1, \omega_2, \dots, \omega_k)$ such that no two entries lie in the same block. We then say $G \leqslant \mathrm{Sym}(\Omega)$ is k**-by-block-transitive** if there are at least k blocks, and G acts transitively on $\Omega^{[k]}$. Note that if G is k-by-block-transitive, it is also k'-by-block-transitive for $k' < k$; in particular, G is transitive on Ω. Also, by considering the orbits of a point stabilizer, it is readily seen (Lemma 2.3) that if G is k-by-block-transitive for some $k \geqslant 2$, then $\sim$ is the coarsest G-invariant equivalence relation other than the universal relation, so $\sim$ can be recovered from the action.

Given a k-by-block-transitive action on a set Ω with equivalence relation $\sim$, we can write Ω as $\Omega_0 \times B$ where the blocks are sets of the form $\{\omega'\} \times B$, and thus consider the action as an imprimitive extension of an action on Ω_0. Clearly, the action on Ω_0 must be k-transitive, that is, transitive on ordered k-tuples of distinct elements. Thus the most basic way to build a k-by-block-transitive action is to form a wreath product $H \wr_{\Omega_0} G_0$, where $G_0 \leqslant \mathrm{Sym}(\Omega_0)$ is k-transitive and $H \leqslant \mathrm{Sym}(B)$ is transitive, and let it act in the natural way on $\Omega_0 \times B$. For the general case of a k-by-block-transitive $G \leqslant \mathrm{Sym}(\Omega)$, then G is equipped with an action $\pi_0 : G \to \mathrm{Sym}(\Omega_0)$ on Ω_0, where $G_0 := \pi_0(G)$ is k-transitive. The point stabilizer $G(\omega)$ is a subgroup of the setwise stabilizer $G([\omega])$ of the block containing ω, and we have $\pi_0(G(\omega)) \leqslant \pi_0(G([\omega]))$. The focus of the present article is to determine the possible values of $\pi_0(G(\omega))$ other than $\pi_0(G([\omega]))$ itself.

In classifying the possibilities for $\pi_0(G(\omega))$, we quickly reduce to the context of **block-faithful** k-by-block-transitive actions, meaning those such that π_0 is injective. Namely, instead

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of considering the original action of G on Ω, it suffices to consider the action of G_0 on the cosets of $L = \pi_0(G(\omega))$. One sees that the action of G_0 on G_0/L is k-by-block-transitive, where the blocks correspond to left cosets of $G_1 = \pi_0(G([\omega]))$. The block size of the resulting action is then the index $|G_1 : L|$.

The purpose of this article is to classify the finite block-faithful k-by-block-transitive permutation groups G for $k \geqslant 2$. For all such groups G, it is enough to specify the group G, the block stabilizer $G_1 := G([\omega])$ and the point stabilizer $L := G(\omega)$. As the action of G on G/G_1 is faithful and k-transitive, we can appeal to the known classification of finite k-transitive permutation groups for $k \geqslant 2$; then all that remains, for each possible pair (G, G_1), is to classify the possible point stabilizers L up to conjugacy, and we can ignore the case $G_1 = L$ as there is nothing new to say here. One sees that in fact G_1 is the unique maximal subgroup of G containing L, so we only need to specify the pair (G, L). The proofs of the results from the introduction will be given in Section 3.8 at the end of the article.

First, let us note that we obtain no interesting examples for $k \geqslant 3$.

Theorem 1.1. *Let $k \geqslant 3$ and let G be a finite block-faithful k-by-block-transitive permutation group. Then G acts k-transitively, that is, the blocks are singletons.*

Corollary 1.2. *Let Ω be a set and let $G \leqslant \mathrm{Sym}(\Omega)$ be k-by-block-transitive, such that $k \geqslant 3$ and the set Ω_0 of blocks is finite. Then for $\omega \in \Omega$ we have $G([\omega]) = KG(\omega)$ where K is the kernel of the action of G on Ω_0.*

For finite block-faithful 2-by-block-transitive actions with nontrivial blocks, the picture is more complicated, but the groups involved are still somewhat special compared to the class of all finite 2-transitive permutation groups. If G has such an action, we find that its socle is one of

$$\mathrm{PSL}_n(q),\ \mathrm{PSU}_3(q),\ {}^2\mathrm{B}_2(q),\ {}^2\mathrm{G}_2(q),\ \mathrm{M}_{11}.$$

Let μ be a generator of $\mathbb{F}_q^*$. Given $g \in \mathrm{PGL}_{n+1}(q)$, we write $\mathrm{Pdet}(g)$ for the set of determinants of matrices representing g: this is a coset of $\det(\mathrm{Z}(\mathrm{GL}_{n+1}(q))) = \langle \mu^{n+1} \rangle$, so we regard $\mathrm{Pdet}(g)$ as an element of the group $\mathbb{F}_q^*/\langle \mu^{n+1} \rangle$. Given $G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$, define

$$\mathrm{Pdet}(G) := \{\mathrm{Pdet}(g) \mid g \in G \cap \mathrm{PGL}_{n+1}(q))\}.$$

Theorem 1.3. *Let G be a finite group with a faithful 2-transitive action on the set Ω_0, extending to a 2-by-block-transitive action of G on the set $\Omega = \Omega_0 \times B$, with block size $|B| \geqslant 2$; let $\omega \in \Omega$. Then G has a nonabelian simple socle S and the stabilizer $G([\omega])$ of the block $[\omega]$ containing ω is the unique maximal subgroup of G that contains $G(\omega)$. If S is of Lie type and naturally represented as a group of (projective) matrices of dimension $n+1$ over $\mathbb{F}_q$, then we can regard G as a subgroup of $\mathrm{P\Gamma L}_{n+1}(q)$, and for $H \leqslant G$ we write $e_H := |H : H \cap \mathrm{PGL}_{n+1}(q)|$. If $S = \mathrm{PSL}_{n+1}(q)$ we take $G([\omega])$ to be a point stabilizer of the usual action of G on the projective n-space $P_n(q)$, and write W for the socle of $G([\omega])$. Up to isomorphism of permutation groups, exactly one of the following is satisfied.*

(a) $\mathrm{PSL}_{n+1}(q) \leqslant G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$, *with $n \geqslant 2$, $q > 2$. In this case $G(\omega)$ contains the soluble residual of $G([\omega])$; if $(n, q) = (2, 3)$, then $G(\omega)$ is of the form $W \rtimes \mathrm{SL}_2(3)$. In addition, $e_{G(\omega)} = e_G$, and the block size $|B|$ divides $q-1$ and is coprime to $|\mathrm{Pdet}(G)|$.*

(b) $\mathrm{PSL}_3(q) \leqslant G \leqslant \mathrm{P\Gamma L}_3(q)$ *and $G(\omega)$ is contained in a group of the form*

$$L^{\Gamma\mathrm{L}_1} = G \cap (W \rtimes \Gamma\mathrm{L}_1(q^2)),\ \textit{such that}\ |L^{\Gamma\mathrm{L}_1} : G(\omega)| \leqslant 2;$$

in this case there are up to three possibilities for the group $G(\omega)$, which belong to different G-conjugacy classes, and $|B| = |L^{\Gamma\mathrm{L}_1} : G(\omega)| q(q-1)/2$.

(c) *S is of rank 1 simple Lie type and the action of G on Ω_0 is the standard 2-transitive action. Moreover, $G(\omega) = N\langle z\rangle$, where N is normal in $G([\omega])$ and contains $G([\omega]) \cap S$, and where $G([\omega])/N$ takes the form $\langle xN\rangle \rtimes \langle zN\rangle$ such that $|\langle xN\rangle| = |\langle zN\rangle| = |B|$. In this case, $|B|$ divides e_G and also divides the order of the multiplicative group of the field.*

(d) *The action is one of eighteen exceptional 2-by-block-transitive actions, listed in Table 1 below.*

The permutation groups described in (a) and (d) are 2-by-block-transitive. In case (b) there are additional constraints on the possible groups $G(\omega)$: see Proposition 3.27. In case (c), the possible block sizes, and the number of conjugacy classes of stabilizers of 2-by-block-transitive actions of the given block size, are both calculated by modular arithmetic: see Proposition 3.21.

For all but the first of the exceptional 2-by-block-transitive actions in Table 1 below, we have

$$\mathrm{PSL}_{n+1}(q) \leqslant G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$$

and $G([\omega])$ is a point stabilizer of the standard action of G on the projective n-space $P_n(q)$; we again write W for the socle of $G([\omega])$. Blank spaces in the table indicate a repeated entry from the line above. The third column indicates the structure of the stabilizer of a distant pair. Note that in many cases $\mathrm{PSL}_3(q) = \mathrm{PGL}_3(q)$ (namely when $q-1$ is not a multiple of 3) or $\mathrm{PGL}_3(q) = \mathrm{P\Gamma L}_3(q)$ (whenever q is prime).

G	$G(\omega)$	$G(\omega, \omega')$	$\|\Omega_0\|$	$\|B\|$
M_{11}	$\mathrm{Alt}(6)$	$C_3^2 \rtimes C_2$	11	2
$\mathrm{PSL}_5(2)$	$W \rtimes \mathrm{Alt}(7)$	$\mathrm{PSL}_3(2)$	31	8
$\mathrm{PSL}_3(5)$	$W \rtimes (\mathrm{SL}_2(3) \rtimes C_4)$	C_4^2	31	5
	$W \rtimes (\mathrm{SL}_2(3) \rtimes C_2)$	C_2^2		10
	$W \rtimes \mathrm{SL}_2(3)$	$\{1\}$		20
$\mathrm{PSL}_3(7)$	$W \rtimes (\mathrm{SL}_2(3).C_2)$	C_3	57	14
$\mathrm{P\Gamma L}_3(7)$	$W \rtimes (\mathrm{SL}_2(3).C_2 \times C_3)$	C_3^2		
$\mathrm{PSL}_3(9)$	$W \rtimes (\mathrm{SL}_2(5).C_4)$	$\mathrm{Sym}(3)^2$	91	12
$\mathrm{P\Gamma L}_3(9)$	$W \rtimes (\mathrm{SL}_2(5).D_8)$	$\mathrm{Sym}(3)^2 \times C_2$		
$\mathrm{PSL}_3(11)$	$W \rtimes (\mathrm{SL}_2(5) \times C_5)$	C_5^2	133	22
	$W \rtimes \mathrm{SL}_2(5)$	$\{1\}$		110
	$W \rtimes (\mathrm{GL}_2(3) \times C_5)$	C_2^2		55
	$W \rtimes (\mathrm{SL}_2(3) \times C_5)$	$\{1\}$		110
$\mathrm{P\Gamma L}_3(19)$	$W \rtimes (\mathrm{SL}_2(5) \times C_9)$	C_3^2	381	114
$\mathrm{PSL}_3(23)$	$W \rtimes (\mathrm{SL}_2(3).C_2 \times C_{11})$	$\{1\}$	553	506
$\mathrm{PSL}_3(29)$	$W \rtimes ((\mathrm{SL}_2(5) \rtimes C_2) \times C_7)$	C_2^2	871	406
	$W \rtimes (\mathrm{SL}_2(5) \times C_7)$	$\{1\}$		812
$\mathrm{PSL}_3(59)$	$W \rtimes (\mathrm{SL}_2(5) \times C_{29})$	$\{1\}$	3541	3422

Table 1: Exceptional 2-by-block-transitive actions

The following emerges as an observation on the classification.

Corollary 1.4. *Let G be a finite block-faithful 2-by-block-transitive permutation group. Then the socle of a block stabilizer acts trivially on that block.*

We can also classify the finite sharply 2-by-block-transitive permutation groups, that is, actions preserving an equivalence relation on a finite set such that for any two distant pairs, there is exactly one element mapping the first distant pair to the second distant pair. With six exceptions, these are sharply 2-transitive or arise from case (b) of Theorem 1.3.

Theorem 1.5. *Let G be a group acting on the finite set Ω and acting faithfully on $\Omega^{[2]}$. Then G acts regularly on $\Omega^{[2]}$ if and only if one of the following holds.*

(a) G is sharply 2-transitive, in other words, the blocks are singletons.

(b) We are in case (b) of Theorem 1.3 with $|L^{\Gamma L_1} : G(\omega)| = 2$ and $|G| = |\mathrm{PGL}_3(q)|$, and the block size is $q(q-1)$.

(c) We have one of the six exceptional actions from Table 1 for which the stabilizer of a distant pair is trivial; in particular, $G = \mathrm{PSL}_3(q) = \mathrm{P\Gamma L}_3(q)$ with $q \in \{5, 11, 23, 29, 59\}$, and the block size is $q(q-1)$.

We conclude this introduction with some remarks on the main theorems and their context in the literature.

Remark 1.6.

(1) The author's original motivation for classifying finite block-faithful 2-by-block-transitive actions is an application to groups acting on infinite locally finite trees. Specifically, if T is a locally finite tree and the closed subgroup $G \leqslant \mathrm{Aut}(T)$ acts 2-transitively on the space of ends ∂T of T, it is not hard to see that each vertex stabilizer $G(v)$ acts 2-by-block-transitively on ∂T, where the blocks correspond to the neighbours of v in T; since there are finitely many blocks, one obtains a quotient 2-by-block-transitive action on a finite set. This application to groups acting on trees is developed in a separate article ([14]).

(2) By taking G up to isomorphism as a permutation group, in the cases that G has socle $\mathrm{PSL}_{n+1}(q)$ for $n \geqslant 2$, we are effectively ignoring the distinction between the set $P_n(q)$ of lines in $\mathbb{F}_q^{n+1}$ and the set $P_n^*(q)$ of n-dimensional subspaces of $\mathbb{F}_q^{n+1}$, which yield isomorphic permutation groups (via the inverse transpose automorphism of G) but not equivalent G-sets (in other words, the corresponding point stabilizers are not conjugate in G).

(3) A previous classification theorem that served as an inspiration for Theorem 1.3 is the classification by Devillers, Giudici, Li, Pearce and Praeger ([4, Theorem 1.2]) of the finite quasiprimitive imprimitive rank 3 permutation groups (where the **rank** is the number of orbits of a point stabilizer). Any such group is easily seen to be 2-by-block-transitive and block-faithful; the three orbits of $G(\omega)$ are $\{\omega\}$, $[\omega]\backslash\{\omega\}$ and $\Omega\backslash[\omega]$, so naturally, the classification in [4] describes a special case of the actions given in Theorem 1.3, namely the case when $G([\omega])$ acts 2-transitively on $G([\omega])/G(\omega)$ (which is automatically the case if $|B| = 2$, but not otherwise; indeed, $G(\omega)$ need not even be maximal in $G([\omega])$). Specifically, the groups in [4, Table 1] relate to Theorem 1.3 as follows:

   (a) Case (a) of Theorem 1.3 includes line 3 of [4, Table 1], but extra conditions are required to ensure the action has rank 3.

   (b) Most actions in case (b) of Theorem 1.3 do not have rank 3, but there are five exceptions, all with $G(\omega) = L^{\Gamma L_1}$. Here G is one of $\mathrm{PGL}_3(4), \mathrm{P\Gamma L}_3(4), \mathrm{P\Gamma L}_3(8), \mathrm{PSL}_3(2), \mathrm{PSL}_3(3)$, listed in lines 4, 5, 8, 9, 10 of [4, Table 1].

   (c) Case (c) of Theorem 1.3 includes line 2 of [4, Table 1], but a rank 3 action only arises if $|B| = 2$, which is limited to actions with socle $\mathrm{PSL}_2(q)$. By contrast, the other types of socle give examples of 2-by-block-transitive actions with odd block size n and rank $n+1$, see Example 3.25 below.

   (d) The first three lines of Table 1 have rank 3, and are listed in lines 1, 7 and 6 of [4, Table 1].

(4) The first line of Table 1 is a notable 'near miss' for an example of a finite block-faithful 3-by-block-transitive action with nontrivial blocks: as well as being 2-by-block-transitive and 4-transitive on blocks, it has only two orbits on distant triples (see Lemma 3.3).

Structure of article The remainder of the article is divided into two sections. In Section 2 we show some basic properties of k-by-block-transitive actions and recall the necessary information about the classification of finite 2-transitive permutation groups. The main section is Section 3, where we work through the classification of finite block-faithful 2-by-block-transitive permutation groups on a case-by-case basis.

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2 Preliminaries

2.1 Generalities on 2-by-block-transitive groups

Here we note some general properties of 2-by-block-transitive groups.

The first thing to note is the following double coset formula for 2-by-block-transitive action. Throughout, we write $[\omega]$ for the $\sim$-class of $\omega \in \Omega$.

Lemma 2.1. *Let Ω be a set, let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive group preserving the nonuniversal equivalence relation $\sim$ and let $\omega \in \Omega$. Then G is 2-by-block-transitive if and only if the following equation is satisfied for some (or equivalently for all) $g \in G \backslash G([\omega])$:*

$$G = G(\omega)gG(\omega) \sqcup G([\omega]).$$

Proof. Since G acts transitively on Ω, there is a bijection θ from the set of $(G(\omega), G(\omega))$-double cosets in G to the set of $G(\omega)$-orbits on Ω, where given $x \in G$ we set

$$\theta(G(\omega)xG(\omega)) = \{h(x\omega) \mid h \in G(\omega)\}.$$

Note that $G([\omega])$ is itself a union of $(G(\omega), G(\omega))$-double cosets, and the $G([\omega])$-orbit of ω is exactly $[\omega]$; thus $\bigcup\{\theta(G(\omega)xG(\omega)) \mid x \in G([\omega])\}$ is the union of all $G(\omega)$-orbits on $[\omega]$.

In particular, the double coset equation in the statement is equivalent to the assertion that $G(\omega)$ acts transitively on $\Omega\backslash[\omega]$. In turn, we see that $G(\omega)$ acts transitively on $\Omega\backslash[\omega]$ if and only if G acts transitively on distant pairs. □

If G is finite, we obtain a formula for the order of the stabilizer of a distant pair.

Corollary 2.2. *Let Ω be a set and let G be a finite group acting transitively on Ω and preserving the equivalence relation $\sim$. Let $(\omega, \omega') \in \Omega^{[2]}$. Then G acts 2-by-block-transitively on Ω if and only if*

$$|G(\omega, \omega')| = \frac{|G(\omega)|^2}{|G| - |G([\omega])|}.$$

In particular, if G acts 2-by-block-transitively on Ω then the right-hand side of the above equation is an integer.

Proof. Since G is transitive we can write $\omega' = g\omega$ for some $g \in G$; note that $g \notin G([\omega])$. We can calculate the size of the double coset $G(\omega)gG(\omega)$ as

$$|G(\omega)gG(\omega)| = |G(\omega)gG(\omega)g^{-1}| = |G(\omega)G(\omega')| = \frac{|G(\omega)||G(\omega')|}{|G(\omega) \cap G(\omega')|} = \frac{|G(\omega)|^2}{|G(\omega, \omega')|},$$

so the equation in the statement is equivalent to the equation

$$|G(\omega)gG(\omega)| = |G| - |G([\omega])|.$$

By Lemma 2.1, the latter equation is equivalent to G acting 2-by-block-transitively on Ω. □

Using the standard correspondence between systems of imprimitivity of a transitive permutation group and subgroups containing a point stabilizer, we deduce the following.

Lemma 2.3. *Let Ω be a set and let $G \leqslant \mathrm{Sym}(\Omega)$ be k-by-block-transitive on Ω relative to the equivalence relation $\sim$, for $k \geqslant 2$. Then for each $\omega \in \Omega$, the block stabilizer $G([\omega])$ is the largest proper subgroup of G containing $G(\omega)$. Equivalently, $\sim$ is the coarsest nonuniversal G-invariant equivalence relation. In particular, we can recover $\sim$ from the action of G on Ω.*

Proof. Since G is k-by-block-transitive, certainly G is transitive and there are $k \geqslant 2$ blocks, so $\sim$ cannot be the universal relation. We see that $\sim$ is G-invariant because if it were not, say $x \sim y$ but $gx \nsim gy$, then we would have a distant k-tuple $(gx, gy, x_3, \dots, x_k)$ in the same G-orbit as the nondistant k-tuple $(x, y, g^{-1}x_3, \dots, g^{-1}x_k)$. In particular, we have $G(\omega) \leqslant G([\omega]) < G$.

Suppose now that H is some subgroup of G containing $G(\omega)$, and suppose that $H \nleqslant G([\omega])$, say $g \in H \backslash G([\omega])$; let O be the H-orbit of $g\omega$. Then $g\omega \in \Omega\backslash[\omega]$; since $H \geqslant G(\omega)$ and $G(\omega)$ acts transitively on $\Omega\backslash[\omega]$ it follows that $\Omega\backslash[\omega] \subseteq O$, in particular $[g\omega] \subseteq O$. But now also $g^{-1}[g\omega] = [\omega] \subseteq O$, so in fact $O = \Omega$, that is, H is transitive on Ω. Since $G(\omega) \leqslant H$, we conclude that $H = G$.

From the previous paragraph, we deduce that $\sim$ is the coarsest G-invariant equivalence relation other than the universal relation. □

Here is a special case, which rules out an affine type action on the set of blocks as soon as the blocks are nontrivial.

Corollary 2.4. *Let Ω be a set, let $G \leqslant \mathrm{Sym}(\Omega)$ be block-faithful and k-by-block-transitive on Ω relative to the nontrivial equivalence relation $\sim$, for $k \geqslant 2$. Let N be a nontrivial normal subgroup of G. Then $G = NG(\omega)$. In particular, N is not abelian.*

Proof. Since G acts k-transitively on the set of blocks, we see that N acts transitively on the set of blocks, so $N \nleqslant G([\omega])$. It then follows by Lemma 2.3 that $G = NG(\omega)$. Since $\sim$ is nontrivial we have $G(\omega) < G([\omega])$, so $N \cap G([\omega])$ is nontrivial. Since N acts faithfully on the set of blocks we deduce that N is not abelian. □

We observe the following property of 2-by-block-transitive actions, which will be used without further comment.

Corollary 2.5. *Let G be a group and let $L < K < G$ be proper subgroups. If the action of G on G/L is 2-by-block-transitive with block stabilizer $M \geqslant L$, then $M \geqslant K$ and the action on G/K is 2-by-block-transitive with block stabilizer M.*

Proof. Suppose G has 2-by-block-transitive action on $\Omega = G/L$ and write ω for the trivial coset. Then $M = G([\omega])$; by Lemma 2.3, since $L < K < G$ we have $K \leqslant M$. Let $g \in G\backslash M$. By Lemma 2.1 we have $G = LgL \cup M$; it then follows that $G = KgK \cup M$, so the action of G on G/K is 2-by-block-transitive by Lemma 2.1. □

Given a group G, we say the G-set Ω is an **extension** of the G-set Ω_0 if there is a G-equivariant surjection π from Ω to Ω_0. The fibres of π then form a system of imprimitivity for G acting on Ω.

Given a k-by-block-transitive permutation group $G \leqslant \mathrm{Sym}(\Omega)$, we have an associated k-transitive action (possibly also k'-transitive for some $k' > k$) $\pi : G \to \mathrm{Sym}(\Omega/\sim)$. We say G is **block-faithful** if G acts faithfully on $\Omega/\sim$. Thus every block-faithful k-by-block-transitive action is an extension of a faithful k-transitive action of the same group.

The next lemma gives a necessary and sufficient condition for a block-faithful extension of a 2-transitive action to be 2-by-block-transitive.

Lemma 2.6. *Let Ω be a set and let G be a group acting on Ω that preserves the nonuniversal equivalence relation $\sim$, acts faithfully and 2-transitively on $\Omega/\sim$ and acts transitively on Ω.*

(i) *Given* $\omega, \omega' \in \Omega$ *with* $\omega \nsim \omega'$, *then there is an involution* $s \in G$ *such that* $s[\omega] = [\omega']$, *and such that*

$$G(\{[\omega], [\omega']\}) = G([\omega], [\omega']) \rtimes \langle s \rangle.$$

(ii) *Let* $\omega \in \Omega$, *let* $L = G(\omega)$ *and let* $s \in G$ *be an involution such that* $s\omega \nsim \omega$. *Then* G *has* 2*-by-block-transitive action on* Ω *if and only if*

$$G([\omega]) = LsL([s\omega])s. \tag{1}$$

In particular, if G *has* 2*-by-block-transitive action on* Ω *then*

$$G([\omega], [s\omega]) = L([s\omega])sL([s\omega])s; \tag{2}$$

if in addition G *is finite, then*

$$|G([\omega], [s\omega]) : L([s\omega])| = |G([\omega]) : L|. \tag{3}$$

Proof. (i) Since G acts 2-transitively on $\Omega/\sim$, there is an element swapping two blocks, so G has an element of even order, and hence an element s of order 2. Since G acts faithfully on $\Omega/\sim$, there is some block not fixed by s, say $s[\omega_0] = s[\omega_0']$ where $\omega_0, \omega_0' \in \Omega$ such that $\omega_0 \nsim \omega_0'$. Since G is 2-transitive on $\Omega/\sim$, after conjugating s by a suitable element of G, we can in fact take $[\omega_0] = [\omega]$ and $[\omega_0'] = [\omega']$. The form taken by $G(\{[\omega], [\omega']\})$ is now clear.

(ii) We have $G([\omega]) = LsL([s\omega])s$ if and only if L acts transitively on the coset space $X = G([\omega])/sL([s\omega])s$. As a $G([\omega])$-space, X is equivalent to the $G([\omega])$-orbit Y of $s\omega$, since

$$sL([s\omega])s = sLs \cap G([\omega]).$$

Since G acts 2-transitively on blocks, we see that Y intersects every block other than $[\omega]$.

If G is 2-by-block-transitive, then $Y = \Omega \backslash [\omega]$ and L acts transitively on $\Omega \backslash [\omega]$, so we deduce (1) from the previous paragraph, and then (2) follows from (1) by intersecting both sides with the group $G([s\omega])$. If G is finite then (3) follows from (2) and the fact that L acts transitively on $(\Omega/\sim) \backslash \{[\omega]\}$, so that $|L : L([s\omega])| = |G([\omega]) : G([\omega], [s\omega])|$.

Conversely, assume (1). Then L acts transitively on Y, so L acts transitively on $(\Omega/\sim) \backslash \{[\omega]\}$. We can write the last statement as a double coset equation, using the fact that $[s\omega] \neq [\omega]$:

$$G = LsG([\omega]) \sqcup G([\omega]);$$

equivalently,

$$G = G([\omega])sL \sqcup G([\omega]),$$

which means that $G([\omega])$ has only two orbits on Ω, one of which is $\Omega \backslash [\omega]$. Thus by (1), in fact L acts transitively on $\Omega \backslash [\omega]$; hence G acts transitively on distant pairs. □

Suppose now we have a 2-transitive action of G on the set Ω, with 2-by-block transitive action on G/L for $L \leqslant G(\omega)$. In classifying the 2-by-block-transitive extensions of the action on Ω, it is sometimes convenient to count candidates for L up to $G(\omega)$-conjugacy, rather than up to G-conjugacy. The next lemma shows that these two ways of counting extensions are equivalent, if we exclude the degenerate case when $|\Omega| \leqslant 2$.

Lemma 2.7. *Let* Ω *be a set with at least* 3 *elements, let* G *be a group acting* 2*-transitively on* Ω, *and let* $L \leqslant G(\omega)$ *be such that* G *acts* 2*-by-block-transitively on* G/L. *Then for all* $g \in G$ *we have* $gLg^{-1} \leqslant G(\omega)$ *if and only if* $g \in G(\omega)$.

Proof. If $g \in G(\omega)$, then clearly $gLg^{-1} \leqslant G(\omega)$.

Conversely, suppose $g \in G$ is such that $gLg^{-1} \leqslant G(\omega)$. Then by Lemma 2.3, $gG(\omega)g^{-1}$ is the largest proper subgroup of G that contains gLg^{-1}, so $G(\omega) \leqslant gG(\omega)g^{-1}$. Since G acts 2-transitively on Ω, the subgroup $G(\omega)$ is already maximal among proper subgroups, so we must have $G(\omega) = gG(\omega)g^{-1}$, that is, $g \in \mathrm{N}_G(G(\omega))$. There are two possibilities for the normalizer: either $\mathrm{N}_G(G(\omega)) = G(\omega)$ or $\mathrm{N}_G(G(\omega)) = G$. In the former case we are done. In the latter case, $G(\omega)$ fixes Ω pointwise, which is incompatible with 2-transitivity given the assumption that $|\Omega| \geqslant 3$. □

Remark 2.8. Based on Lemma 2.1 one can classify the 2-by-block-transitive actions of a given finite group G by the following simple (but not particularly efficient) procedure:

(1) Determine a set $\mathcal{M}$ of representatives of the conjugacy classes of subgroups H of G such that G acts 2-transitively on G/H.

(2) For each $H \in \mathcal{M}$, fix $g_H \in G \backslash H$.

(3) We produce a sequence $\mathcal{L}_i^H$ of sets of subgroups of H, starting with $\mathcal{L}_0^H = \{H\}$. For each $L \in \mathcal{L}_i^H$ we take representatives of the H-conjugacy classes occurring as maximal subgroups M of L; then for each representative M, we check if $G = Mg_HM \cup H$ (for instance by calculating whether $|Mg_HM| + |H| = |G|$). We then form a set $\mathcal{L}_{i+1}^H$ consisting of those M in the previous sentence such that $G = Mg_HM \cup H$.

(4) The previous step eventually terminates; write $\mathcal{L}^H = \bigcup_{i \geqslant 0} \mathcal{L}_i^H$.

(5) Write $\mathcal{L} = \bigcup_{H \in \mathcal{M}} \mathcal{L}^H$. Up to equivalence, the 2-by-block-transitive actions of G are given by the action on G/L for $L \in \mathcal{L}$.

2.2 Finite 2-transitive permutation groups of almost simple type

The finite 2-transitive permutation groups are all known; an overview can be found for example in [5, §7.7]. Every such group is either of affine type or has nonabelian simple socle, and given Corollary 2.4 we can focus on the latter. These groups are displayed in the next table, with the rows corresponding to the action of the socle S, where two actions are identified if their stabilizers belong to the same $\mathrm{Aut}(S)$-class. We also indicate the degree t of transitivity, the largest overgroup $N := \mathrm{N}_{\mathrm{Sym}(\Omega)}(S)$ of the socle compatible with the action and the structure of a point stabilizer of N; where applicable, P denotes a normal subgroup of S that acts regularly on $\Omega \backslash \{\omega\}$. In all cases the socle itself acts 2-transitively, except for the action of $\mathrm{P\Gamma L}_2(8) = {}^2\mathrm{G}_2(3)$ on 28 points.

Degree	t	S	N	$N(\omega)$
$d \geqslant 5$	$d-2$ or d	$\mathrm{Alt}(d)$	$\mathrm{Sym}(d)$	$\mathrm{Sym}(d-1)$
$^{a}\frac{q^{n+1}-1}{q-1}$	2 or 3	$\mathrm{PSL}_{n+1}(q)$	$\mathrm{P\Gamma L}_{n+1}(q)$	$(C_p^{en} \rtimes \mathrm{GL}_n(q)) \rtimes \langle \phi \rangle$
$q^3+1, q \geqslant 3$	2	$\mathrm{PSU}_3(q)$	$\mathrm{P\Gamma U}_3(q)$	$P \rtimes \langle x, \phi \rangle$
$q^2+1, q=2^{2n+1}>2$	2	$^2\mathrm{B}_2(q)$	$^2\mathrm{B}_2(q) \rtimes \langle \phi \rangle$	$P \rtimes \langle x, \phi \rangle$
$q^3+1, q=3^{2n+1}>3$	2	$^2\mathrm{G}_2(q)$	$^2\mathrm{G}_2(q) \rtimes \langle \phi \rangle$	$P \rtimes \langle x, \phi \rangle$
$2^{2n+1}-2^n$	2	$\mathrm{Sp}_{2n+2}(2)$	$\mathrm{Sp}_{2n+2}(2)$	$\Omega^-_{2n+2}(2).C_2$
$2^{2n+1}+2^n$	2	$\mathrm{Sp}_{2n+2}(2)$	$\mathrm{Sp}_{2n+2}(2)$	$\Omega^+_{2n+2}(2).C_2$
11	2	$\mathrm{PSL}_2(11)$	$\mathrm{PSL}_2(11)$	$\mathrm{Alt}(5)$
11	4	M_{11}	M_{11}	$\mathrm{Alt}(6).C_2$
12	3	M_{11}	M_{11}	$\mathrm{PSL}_2(11)$
12	5	M_{12}	M_{12}	M_{11}
15	2	$\mathrm{Alt}(7)$	$\mathrm{Alt}(7)$	$\mathrm{PSL}_3(2)$
22	3	M_{22}	$\mathrm{M}_{22} \rtimes C_2$	$\mathrm{PSL}_3(4) \rtimes C_2$
23	4	M_{23}	M_{23}	M_{22}
24	5	M_{24}	M_{24}	M_{23}
28	2	$\mathrm{PSL}_2(8)$	$\mathrm{P\Gamma L}_2(8)$	$C_9 \rtimes C_6$
176	2	HS	HS	$\mathrm{PSU}_3(5) \rtimes C_2$
276	2	Co_3	Co_3	$\mathrm{McL} \rtimes C_2$

[a] $(n,q) \notin \{(1,2),(1,3),(1,4)\}$

2.3 Transitive semilinear groups

For the classification of finite block-faithful 2-by-block-transitive actions, we will also need some aspects of the classification of finite 2-transitive affine groups, which are summarized in the following lemma.

Lemma 2.9 (Hering [8]; see also [12, Appendix 1]). *Let $H = V \rtimes G$ be a finite 2-transitive affine permutation group, where V is regarded as the additive group of some vector space on which G acts by semilinear maps. Then G and V can be taken as follows:*

(i) V is the field of order q, and $G \leqslant \Gamma\mathrm{L}_1(q)$;

(ii) V is the vector space of dimension n over the field of order q, and $\mathrm{SL}_n(q) \trianglelefteq G$;

(iii) V is the vector space of dimension $2m$ over the field of order q, and $\mathrm{Sp}_{2m}(q) \trianglelefteq G$;

(iv) V is the vector space of dimension 6 over the field of order $q = 2^e$, and $\mathrm{G}_2(q) \trianglelefteq G$;

(v) H is one of finitely many exceptional 2-transitive groups of affine type, with V being a vector space of dimension 2, 4 or 6. In each case, either H is soluble or H has a unique nonabelian composition factor, which is one of: $\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{Alt}(7), \mathrm{PSL}_2(13)$. *If V has dimension 2, then $q \in \{5, 7, 9, 11, 19, 23, 29, 59\}$.*

3 2-by-block-transitive groups

In this section we will obtain a classification of the block-faithful k-by-block-transitive actions of finite groups, as promised in the introduction. For the most part, the proof will be via case analysis of the finite 2-transitive permutation groups.

Let us set some notation for linear and projective spaces that will be used throughout this section.

Definition 3.1. Let p be a prime and $q = p^e$ for some positive integer e. We give the vector space $V = \mathbb{F}_q^{n+1}$ a standard basis $\{v_0, \dots, v_n\}$. Given a subset X of V, write $\langle X \rangle_q$ for the $\mathbb{F}_q$-subspace of V generated by X and write $\alpha_i = \langle v_i \rangle_q$. We write $P_n(q)$ for the set of lines in $\mathbb{F}_q^{n+1}$. Write $\Gamma\mathrm{L}_{n+1}(q)$ for the group of semilinear maps from $\mathbb{F}_q^{n+1}$ to itself. In this context there is an element $\phi \in \Gamma\mathrm{L}_{n+1}(q)$ of order e that acts by sending $\sum_{i=0}^{n} \lambda_i v_i$ to $\sum_{i=0}^{n} \lambda_i^p v_i$; we will refer to $\Gamma\mathrm{L}_{n+1}(q)$-conjugates of powers of ϕ as **field automorphisms**. One can also consider ϕ to act as an automorphism of $\mathrm{GL}_{n+1}(q)$ that transforms matrices by sending every entry to its p-th power; however, it is important to note that the action of $\mathrm{GL}_{n+1}(q)$ on V also naturally extends to an action of $\Gamma\mathrm{L}_{n+1}(q) = \mathrm{GL}_{n+1}(q) \rtimes \langle \phi \rangle$. (By contrast, for $n \geqslant 2$ the natural action of $\mathrm{GL}_{n+1}(q)$ on V cannot be extended to incorporate the inverse transpose automorphism of $\mathrm{GL}_{n+1}(q)$.)

There is an induced action of $\Gamma\mathrm{L}_{n+1}(q)$ on $P_n(q)$, with kernel the scalar matrices; we write $\mathrm{P}\Gamma\mathrm{L}_{n+1}(q)$ for $\Gamma\mathrm{L}_{n+1}(q)$ modulo the scalar matrices. Note that $\mathrm{P}\Gamma\mathrm{L}_{n+1}(q)$ acts (at least) 2-transitively on $P_n(q)$.

In this section, some calculations on individual finite groups, namely to determine subgroups of given indices and enumerate double cosets (the latter to determine if an action is 2-by-block-transitive) were performed using the computer algebra package GAP [7]. We omit the details of these routine computations. The author also used the online ATLAS of Finite Group Representations [1] as an indicative reference for some properties of finite groups, however it is not required for the proofs.

3.1 Some specific 2-transitive groups

We start with some groups that are convenient to deal with individually.

Lemma 3.2. *Let G be one of*

$$\mathrm{PSL}_2(11), \mathrm{Alt}(7), \mathrm{P}\Gamma\mathrm{L}_2(8), \mathrm{HS}, \mathrm{Co}_3$$

acting 2-transitively on a set X of d points, where $d = 11, 15, 28, 176, 276$ respectively. Then X does not extend properly to a 2-by-block-transitive action of G.

Proof. Let x and y be distinct elements of X and let s be an involution of G such that $sx = y$ and $sy = x$, as in Lemma 2.6(i). Let $L \leqslant G(x)$ be a point stabilizer of a 2-by-block-transitive action of G.

For $G = \mathrm{PSL}_2(11)$ acting on 11 points, we see that $|G(x,y)| = 6$; for $G = \mathrm{P}\Gamma\mathrm{L}_2(8)$ acting on 28 points then $|G(x,y)| = 2$. For $G = \mathrm{HS}$ acting on 176 points, we have $G(x) = \mathrm{PSU}_3(5) \rtimes C_2$ and $G(x,y) = \mathrm{Aut}(\mathrm{Alt}(6))$, so every automorphism of $G(x,y)$ is inner. In all of these cases, we see that there is no proper subgroup H of $G(x,y)$ such that $G(x,y) = HsHs$. The conclusion now follows by Lemma 2.6.

For $G = \mathrm{Alt}(7)$ acting on 15 points, we have $|G| - |G(x)| = 2^4 \cdot 3 \cdot 7^2$, so by Corollary 2.2 we would need $|L|$ to be a multiple of $2^2 \cdot 3 \cdot 7$, or in other words, index at most 2 in $G(x)$. However, $G(x) \cong \mathrm{PSL}_3(2)$ is simple and therefore has no subgroup of index 2. Thus there is no proper extension of X to a 2-by-block-transitive action in this case.

For $G = \mathrm{Co}_3$ acting on 276 points, we have $G(x) = \mathrm{McL} \rtimes C_2$ and $G(x,y) = \mathrm{PSU}_4(3) \rtimes C_2$. In order to achieve $G(x,y) = L(y)sL(y)s$, as in Lemma 2.6, we would need $G(x,y) = AsAs$ for some maximal subgroup A of $G(x,y)$ containing $L(y)$. Given that $G(x,y)$ is almost simple, one sees from [2, Theorem 1.1] that no such A exists. $\square$

Lemma 3.3. *Among the Mathieu groups (including $\mathrm{M}_{22} \rtimes C_2$) there is only one proper 2-by-block-transitive action, namely the imprimitive rank 3 action of M_{11}, which is given in line 1 of Table 1. The latter action has exactly two orbits on distant triples, each of size 3960.*

Proof. We suppose that G is a Mathieu group equipped with one of its multiply transitive actions on the set X, take $x \in X$, and suppose $L < G(x)$ is such that the action on G/L is 2-by-block-transitive. We again use the fact given by Corollary 2.2 that $|L|^2$ must be a multiple of $|G| - |G(x)|$.

First consider $(G, G(x)) = (\mathrm{M}_{11}, \mathrm{M}_{10})$. In this case $|G| - |G(x)| = 2^5 \cdot 3^2 \cdot 5^2$, so we need $|L|$ to be a multiple of $2^3 \cdot 3 \cdot 5 = 120$. We find that the only subgroup of $G(x) = \mathrm{M}_{10}$ of sufficient order is the subgroup $\mathrm{Alt}(6)$ of order 2. In this case we indeed obtain a 2-by-block-transitive action of M_{11} (indeed an imprimitive rank 3 action, see [4]) on the 22-point set $\Omega = \mathrm{M}_{11}/\mathrm{Alt}(6)$, where the blocks correspond to cosets of M_{10}. The stabilizer of a distant pair (ω_1, ω_2) is a group $H = C_3^2 \rtimes C_2$ of order 18. A calculation in GAP finds that the action of H on Ω has four fixed points (which must be the points in $[\omega_1] \cup [\omega_2]$) plus two orbits of size 9. Thus M_{11} has two orbits on distant triples, each of size $9 \cdot 22 \cdot 20 = 3960$.

Next suppose $|X| = 12$, so G is either M_{11} with point stabilizer $\mathrm{PSL}_2(11)$, or M_{12} in its natural action. In either case, $|G| - |G(x)|$ is a multiple of 22, so $|L|$ is also a multiple of 22; however, the only proper subgroups of M_{11} of order a multiple of 22 are the point stabilizers of the 3-transitive action of M_{11} on 12 points. This rules out any proper 2-by-block-transitive actions when $G = \mathrm{M}_{11}$, and for $G = \mathrm{M}_{12}$ one can check by double coset enumeration that M_{12} does not in fact act 2-by-block-transitively on the cosets of $\mathrm{PSL}_2(11)$.

Finally we suppose G is one of the large Mathieu groups, including $\mathrm{M}_{22} \rtimes C_2$, acting on $22 \leqslant d \leqslant 24$ points. In each case $|G| - |G(x)|$ is a multiple of $2^6 \cdot 5 \cdot p$ where $p \in \{7, 11, 23\}$ is the largest prime dividing $d - 1$, so $|L|$ must be divisible by $2^3 \cdot 5 \cdot p$. After checking the orders of maximal subgroups of $G(x)$, we are only left with $G = \mathrm{M}_{22} \rtimes C_2$, $G(x) = \mathrm{PSL}_3(4) \rtimes C_2$ and $L \leqslant \mathrm{PSL}_3(4)$. This remaining case is ruled out by Corollary 2.4, since we would have L contained in the proper normal subgroup M_{22} of G. □

Lemma 3.4. *Let $G = \mathrm{PSL}_5(2)$. Then there is exactly one $\mathrm{Aut}(G)$-conjugacy class of subgroups L of G such that G has proper 2-by-block-transitive action on G/L, which is the one indicated in line 2 of Table 1.*

Proof. We first note that G has only one 2-transitive action up to conjugacy in $\mathrm{Aut}(G)$, namely the standard action of G on the projective space $P_4(2)$, so we only need to consider subgroups $L \leqslant G(v)$ for some fixed $v \in P_4(2)$. Since $|G| - |G(v)|$ is a multiple of 7, by Corollary 2.2 we may assume $|L|$ is a multiple of 7.

Let $\mathcal{L}$ be the set of proper subgroups of L of $G(v)$ that have both of the following properties: $|L|$ is a multiple of 7, and L acts transitively on $P_4(2) \backslash \{v\}$. Calculations in GAP reveal that $\mathcal{L}$ is a single $G(v)$-conjugacy class; given $L_1 \in \mathcal{L}$ we can write $L_1 = W \rtimes \mathrm{Alt}(7)$, where $W = C_2^4$ is the socle of $G(v)$.

One can check (also it is shown in [4]) that L_1 has only three double cosets in G, namely, L_1 itself: the nontrivial double coset inside $G(v)$; and the remaining double coset is $G \backslash G(v)$. In particular, G has 2-by-block-transitive action on G/L_1 by Lemma 2.1. By the previous two paragraphs, up to $\mathrm{Aut}(G)$-conjugacy this is the only proper 2-by-block-transitive action of G. The stabilizer of a distant pair in this action is a group of the form $\mathrm{PSL}_3(2)$. □

3.2 Families of 2-transitive actions admitting no proper 2-by-block-transitive extensions

For some infinite families of 2-transitive actions, we can deduce from known results that there are no proper 2-by-block-transitive extensions.

Lemma 3.5. *Let $\mathrm{Alt}(X) \leqslant G \leqslant \mathrm{Sym}(X)$, where $X = \{1, 2, \dots, d\}$, $2 \leqslant d < \infty$, and let s be an involution in $\mathrm{Sym}(X)$. Then there is no proper subgroup H of G such that $G = HsHs$.*

Proof. Let $H < G$. If $G = \mathrm{Sym}(X)$ or $d \leqslant 4$, then sHs is conjugate to H in G and the conclusion is clear, so we may assume $G = \mathrm{Alt}(X)$ and $d \geqslant 5$. It also suffices to consider the case that H is a maximal subgroup of G. Suppose $G = HsHs$. Then sHs is also maximal in G and isomorphic to H as a permutation group. However, one sees from the two cases of [13, §1, Corollary 5] that it is not possible to write $G = AB$ where A and B are permutationally isomorphic maximal subgroups of G. This contradiction proves the lemma. □

Remark 3.6. If instead of conjugating by a transposition, we applied an automorphism θ of $G = \mathrm{Sym}(d)$ or $G = \mathrm{Aut}(d)$, we would have the following examples of $H < G$ such that $G = H\theta(H)$: if θ represents the exotic outer automorphism of G where G is $\mathrm{Sym}(6)$ or $\mathrm{Alt}(6)$, then a point stabilizer H would satisfy $G = H\theta(H)$, since the action of H on $G/\theta(H)$ corresponds to the action of $\mathrm{Sym}(5) \cong \mathrm{PGL}_2(5)$ or $\mathrm{Alt}(5) \cong \mathrm{PSL}_2(5)$ on the projective line.

Corollary 3.7. *The natural actions of symmetric and alternating groups do not extend properly to 2-by-block-transitive actions.*

Proof. Let G be the symmetric or alternating group of degree $d \geqslant 2$, acting on $X = \{1, \dots, d\}$. By Lemma 2.6, in order for G to have 2-by-block-transitive action on G/L, a proper subgroup $L < G(1)$ must satisfy

$$G(1, 2) = L(2)sL(2)s,$$

with $H = L(2)$ being a proper subgroup of $G(1, 2)$ and $s \in G$ an involution such that $s(1) = 2$. However $G(1, 2)$ is a symmetric or alternating group, so by Lemma 3.5, no suitable subgroup H exists. ☐

Lemma 3.8. *Let $G = \mathrm{Sp}_{2m}(2)$ in one of its 2-transitive actions, where $m \geqslant 3$. Then the setwise stabilizer of an unordered pair of points $\{\omega, \omega'\}$ splits as a direct product*

$$G(\{\omega, \omega'\}) = G(\omega, \omega') \times \langle s \rangle,$$

where s is an involution swapping ω and ω'. Consequently there is no proper extension of the action to a 2-by-block-transitive action of G.

Proof. We follow the description of the 2-transitive actions of G given in [5, §7.7].

Take a vector space V over $\mathbb{F}_2$ with basis $\{v_1, \dots, v_m, w_1, \dots, w_m\}$; let ψ be the bilinear form such that

$$\psi(v_i, w_j) = \delta_{ij},\ \psi(v_i, v_j) = \psi(w_i, w_j) = \psi(w_i, v_j) = 0,$$

and let G be the subgroup of $\mathrm{GL}(V)$ preserving the nondegenerate symplectic form

$$\varphi : (u, v) \mapsto \psi(u, v) - \psi(v, u).$$

Write Ω for the set of functions $\theta : V \to \mathbb{F}_2$ such that

$$\forall u, v \in V : \varphi(u, v) = \theta(u + v) - \theta(u) - \theta(v).$$

Equivalently, Ω consists of quadratic forms on V that can be written as $\theta_a : u \mapsto \psi(u, u) + \varphi(u, a)$ for $a \in V$.

We now have an action of G on Ω given by $g.\theta_a = \theta_{ga}$, or equivalently $g.\theta_a(u) = \theta_a(g^{-1}u)$. We find that Ω splits into two G-orbits

$$\Omega_+ := \{\theta_a \mid \psi(a, a) = 0\} \text{ and } \Omega_- := \{\theta_a \mid \psi(a, a) = 1\},$$

The actions of G on Ω_+ and Ω_- are both faithful and represent the two standard 2-transitive actions of G, see [5, Theorem 7.7A]. Thus as a permutation group we can take G to be given by its action on Ω_ϵ with $\epsilon \in \{+, -\}$.

Given $a \in V$, write $a^\perp = \{b \in V \mid \varphi(b, a) = 0\}$. Then $a^\perp$ is a subspace of V; moreover, since φ is nondegenerate, if $a \neq 0$ then $a^\perp$ has codimension 1, and if $a \neq b$ then $a^\perp \neq b^\perp$. Given $a \in V \backslash \{0\}$ we claim that the pointwise fixator K_a of $a^\perp$ in G is cyclic of order 2, namely $K_a = \langle t_a \rangle$ where $t_a : u \mapsto u + \varphi(u, a)a$. On the one hand it is clear that t_a fixes $a^\perp$ pointwise, and it is easy to check that t_a is an involution in G ([5, Exercise 7.7.5]). On the other hand, given $g \in K_a$ then $gu - u$ is constant as u ranges over the nontrivial coset of $a^\perp$, so we can write $g : u \mapsto u + \varphi(u, a)c$ for some $c \in V$. Since $g \in G$,

$$\forall u, v \in V : \varphi(u + \varphi(u, a)c, v + \varphi(v, a)c) = \varphi(u, v),$$

in other words

$$\forall u, v \in V : \varphi(u,a)\varphi(v,c) = \varphi(v,a)\varphi(u,c),$$

so $c \in \{0, a\}$ and $g \in \langle t_a \rangle$.

We can take our point stabilizer of the action of G on Ω_ϵ to be the stabilizer $G(\theta_a)$ of the quadratic form $\theta_a : u \mapsto \psi(u,u) + \varphi(u,a)$, such that $\theta_a \in \Omega_\epsilon$. Let θ_b be some other point in Ω_ϵ. Given $g \in G(\theta_a, \theta_b)$, then g fixes $\theta_a + \theta_b$, from which we obtain the equation

$$\forall u \in V : \varphi(u, a+b) = \varphi(g^{-1}u, a+b),$$

so $G(\theta_a, \theta_b)$ preserves the subspace $W = (a+b)^\perp$. By [5, Lemma 7.7A] we know that $s = t_{a+b}$ swaps θ_a and θ_b, so it does not belong to $G(\theta_a, \theta_b)$. In particular, $G(\theta_a, \theta_b) \cap K_{a+b} = \{1\}$, so $G(\theta_a, \theta_b)$ acts faithfully on W and hence commutes with s. Thus the setwise stabilizer $G(\{\theta_a, \theta_b\})$ takes the form $G(\theta_a, \theta_b) \times \langle s \rangle$.

Given $L \leqslant G(\theta_a)$ such that G has 2-by-block-transitive action on G/L, we see by Lemma 2.6 that

$$G(\theta_a) = LsL(\theta_b)s;$$

however, in the present situation, $LsL(\theta_b)s = L$. Thus there is no proper extension of Ω_ϵ to a 2-by-block-transitive action of G. □

3.3 Projective space actions

The next case we need to consider is faithful 2-by-block-transitive actions of groups G with socle $\mathrm{PSL}_{n+1}(q)$ for $n \geqslant 2$ and point stabilizer L, such that the block stabilizer $G_1 \geqslant L$ is a point stabilizer of the standard action of G on projective n-space $P_n(q)$. (We will deal with the case that G has socle $\mathrm{PSL}_2(q)$ separately; socle $\mathrm{PSL}_3(q)$ also brings up some complications that we will deal with later.) Since there is no loss of generality in doing so, we will in fact work with groups $Z\mathrm{SL}_{n+1}(q) \leqslant G \leqslant \Gamma\mathrm{L}_{n+1}(q)$, where Z is the group of scalar matrices in $\mathrm{GL}_{n+1}(q)$.

Let us set some hypotheses and notation for this subsection, which will also be reused later.

Hypothesis 3.9. Let $n \geqslant 2$, let p be a prime, let $q = p^e$ for some $e \geqslant 1$, let $Z\mathrm{SL}_{n+1}(q) \leqslant G \leqslant \Gamma\mathrm{L}_{n+1}(q)$, where Z is the group of scalar matrices in $\mathrm{GL}_{n+1}(q)$. Let μ be a generator of $\mathbb{F}_q^*$. Given $H \leqslant \Gamma\mathrm{L}_{n+1}(q)$ write $H_{\mathrm{GL}} := H \cap \mathrm{GL}_{n+1}(q)$, and write $e_H := |H : H_{\mathrm{GL}}|$. Then we note that that e_H is the largest order of field automorphism in $H\mathrm{GL}_{n+1}(q)$.

Let G act on the standard $(n+1)$-dimensional space $V = \mathbb{F}_q^{n+1}$, with $P_n(q)$ the corresponding projective n-space, and let W be the group of elements of $\mathrm{SL}_{n+1}(q)$ that fix pointwise the spaces V/α_0 and α_0. Write $\mathrm{GL}_n(q)$ for the subgroup of elements of $\mathrm{GL}_{n+1}(q)$ that fix the subspace $\alpha_0 = \langle v_0 \rangle_q$ pointwise and preserve the subspace $\langle v_1, \ldots, v_n \rangle_q$, and write $\mathrm{GL}_1(q)$ for the subgroup of elements of $\mathrm{GL}_{n+1}(q)$ that fix the subspace $\langle v_1, \ldots, v_n \rangle_q$ pointwise and preserve α_0. Take the subgroup $\mathrm{SL}_n(q)$ of elements of $\mathrm{GL}_n(q)$ of determinant 1, and write $M = W \rtimes Z\mathrm{SL}_n(q)$. We see that the linear part of $G(\alpha_0)$ satisfies

$$M \trianglelefteq G(\alpha_0)_{\mathrm{GL}} \leqslant W \rtimes (\mathrm{GL}_n(q) \times \mathrm{GL}_1(q)).$$

Let s be the element of $\mathrm{SL}_{n+1}(q) \leqslant G$ that swaps v_0 and v_1, sends v_2 to $-v_2$ and fixes $v_3, \ldots, v_n$. Thus we have

$$G(\{\alpha_0, \alpha_1\}) = G(\alpha_0, \alpha_1) \rtimes \langle s \rangle.$$

Given $g \in \mathrm{GL}_{n+1}(q)$, we define $\mathrm{Pdet}(g) = \det(gZ)$, regarded as an element of $\mathbb{F}_q^*/\langle \mu^{n+1} \rangle$.

Let $Z \leqslant L \leqslant G(\alpha_0)$; we will be considering for which L the action of G on G/L is 2-by-block-transitive. In anticipation of appealing to Lemma 2.6 we number the following equations for reference, which may or may not be satisfied:

$$G(\alpha_0) = LsL(\alpha_1)s; \tag{4}$$

$$G(\alpha_0, \alpha_1) = L(\alpha_1)sL(\alpha_1)s. \tag{5}$$

Definition 3.10. Under Hypothesis 3.9, we will say the action of G on G/L is a **projective-determinant (PD)** action if it is 2-by-block-transitive and $M \leqslant L \leqslant G(\alpha_0)$.

If $n = 2$, we will say the action of G on G/L is **quadratic-extended projective plane type (QP)** if it is 2-by-block-transitive, we have $Z \leqslant L \leqslant G(\alpha_0)$, and writing $G(\alpha_0) \leqslant W \rtimes \mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$ as before, then LW/W normalizes a Singer cycle in $\mathrm{GL}_2(q)W/W$. (See for instance [9, §2] for some basic information about Singer cycles in linear groups.) Equivalently,

$$Z \leqslant L \leqslant WZ\Gamma\mathrm{L}_1(q^2) \cap G(\alpha_0),$$

where $\Gamma\mathrm{L}_1(q^2)$ acts in the natural way on $\langle v_1, v_2\rangle_q$ (with respect to some $\mathbb{F}_{q^2}$-field structure compatible with the $\mathbb{F}_q$-vector space structure). The terminology will be motivated later, see Example 3.26.

Note that the group $WZ\Gamma\mathrm{L}_1(q^2) \cap G(\alpha_0)$ is specified up to $G(\alpha_0)$-conjugacy. Specifically, WZ is normal in $G(\alpha_0)$ (since Z is central and WZ/Z is the socle of $G(\alpha_0)/Z$), whereas $\Gamma\mathrm{L}_1(q^2)WZ/WZ$ is the normalizer in $\Gamma\mathrm{L}_3(q)(\alpha_0)/WZ \cong \Gamma\mathrm{L}_2(q)$ of a Singer cycle; the Singer cycle is unique up to $\mathrm{GL}_2(q)$-conjugacy, and the conjugation action of $\mathrm{N}_{G(\alpha_0)}(\mathrm{GL}_2(q))$ accounts for all inner automorphisms of $\mathrm{GL}_2(q)$.

If the action of G on G/L is 2-by-block-transitive, but neither projective-determinant nor QP, we will say it is **exceptional**.

We carry over the terminology of PD, QP and exceptional actions to the quotient G/Z in the obvious way.

We first note that PD actions of G/Z are never sharply 2-by-block-transitive. For some later results it will also be useful to record the structure of the stabilizer of a pair of lines and of the group $M \cap sMs$.

Lemma 3.11. *Assume Hypothesis 3.9 and write* $\overline{G} = \langle G, \mathrm{GL}_{n+1}(q)\rangle$.

(i) Let W_i be the kernel of the action of $H := \overline{G}(\alpha_0, \alpha_1)$ on $(V/\alpha_i) \oplus \alpha_i$. Then W_i is a normal subgroup of H of order q^{n-1}, with $W_0 = W(\alpha_1)$. We then have an $\langle s\rangle$-invariant normal subgroup $Z^* = W_0 \times W_1 \times Z$ of H_{GL}, and we have

$$H = (Z^* \rtimes (K \rtimes \langle h\rangle)) \rtimes \langle y_0'\rangle,$$

where K is a copy of $\mathrm{GL}_{n-1}(q)$ acting on $\langle v_2, \dots, v_n\rangle_q$ and fixing v_0 and v_1; h is the element of H_{GL} that sends v_1 to μv_1 and v_2 to $\mu^{-1}v_2$, and fixes v_0 and $v_3, \dots, v_n$; and y_0' is a field automorphism of order e_G.

(ii) We have

$$M \cap sMs = Z^* \rtimes (\mathrm{SL}_{n-1}(q) \rtimes \langle h^{a_0}\rangle),$$

where $a_0 = (q-1)/\gcd(q-1, n+1)$. In particular,

$$|M \cap sMs : Z| = q^{2(n-1)}\gcd(q-1, n+1)|\mathrm{SL}_{n-1}(q)|.$$

(iii) If G has a PD action on G/L, then the action of G/Z on G/L is not sharply 2-by-block-transitive.

Proof. (i) Let $y_0' = \phi^{e/e_G}$ where ϕ is the standard Frobenius automorphism. We see that $\overline{G} = \mathrm{GL}_{n+1}(q) \rtimes \langle y_0'\rangle$ and hence $H = H_{\mathrm{GL}} \rtimes \langle y_0'\rangle$. From now on, let us focus on H_{GL}.

It is clear from the definitions that W_0 and W_1 are normal in H, that $W_0 = W(\alpha_1)$, and that $W_1 = sW_0s$. Since the W-orbit of α_1 has size q, we have $|W_i| = |W|/q = q^{n-1}$. From the definitions we see that $W_0 \cap W_1$ acts trivially on V and hence is trivial; moreover, $Z \cap W_0W_1 = \{1\}$, so we obtain an $\langle s\rangle$-invariant normal subgroup $Z^* = W_0 \times W_1 \times Z$ of H contained in H_{GL}. We can split H_{GL} as a semidirect product $Z^* \rtimes C$ by taking C to be the group of matrices that fix v_0 and stabilize the spaces α_1 and $\langle v_2, \ldots, v_n\rangle_q$. It is then clear that C can be decomposed as $\mathrm{GL}_{n-1}(q) \rtimes \langle h\rangle$ as described. (The choice of h here is made in order to obtain a cyclic complement to K in C that acts with determinant 1 on V/α_0.)

(ii) Since Z^* is $\langle s\rangle$-invariant we have $Z^* \leqslant M \cap sMs$, and it is also clear that $M \cap sMs$ contains all elements of K of determinant 1. It remains to describe $K' := M \cap sMs \cap (K \times \langle h\rangle)$.

Let $g \in K'$. Then $g = kh^a$ for some $k \in K$ and $a \in \mathbb{Z}$, so g fixes $\langle v_0\rangle_q$ and acts as multiplication by μ^a on $\langle v_1\rangle_q$. Since $g \in M \cap sMs$, we also have $g = z_0 g_0 = z_1 g_1$, where $z_0, z_1 \in Z$ and g_i acts with determinant 1 on α_i and V/α_i. In particular, we see that $z_0 = 1$ and z_1 is scalar multiplication by μ^a. Since g and h^a both act with determinant 1 on V/α_0, we see that $\det(k) = 1$. At the same time, calculating the determinant of g on V/α_1 in two ways gives

$$\mu^{-a} = \mu^{an},$$

so $\mu^{a(n+1)} = 1$, in other words, μ^a is a power of μ^{a_0}. Thus $g \in \mathrm{SL}_{n-1}(q) \rtimes \langle h^{a_0}\rangle$.

On the other hand, we have $h \in M$, since h acts with determinant 1 on α_0 and on V/α_0. At the same time, the action of h^{a_0} on α_1 has determinant $\mu^{a_0(n+1)} = 1$, and similarly on V/α_1; thus $h^{a_0} \in M \cap sMs$. We deduce that $K' = \mathrm{SL}_{n-1}(q) \rtimes \langle h^{a_0}\rangle$. The value of $|M \cap sMs : Z|$ is now clear.

(iii) is clear from the fact that $|M \cap sMs : Z| > 1$. □

Our next aim is to show that if $n \geqslant 3$, all 2-by-block-transitive actions are PD, other than when $G = \mathrm{PSL}_5(2)$, a special case that has already been dealt with.

Proposition 3.12. *Assume Hypothesis 3.9. Let $Z \leqslant L \leqslant G(\alpha_0)$ be such that G has 2-by-block-transitive action on G/L, with block stabilizer $G(\alpha_0) \geqslant L$. Then $W \leqslant L$ and one of the following holds:*

(a) We have $M \leqslant L$, in other words, the action is PD.

(b) $n = 2$ and the action of L on $(V/\alpha_0) - \{0\}$ is a transitive subgroup of $\Gamma\mathrm{L}_2(q)$ that does not contain $\mathrm{SL}_2(q)$.

(c) $n = 4$, $q = 2$ and L is the subgroup $C_2^4 \rtimes \mathrm{Alt}(7)$ of $G = \mathrm{PSL}_5(2)$ given in Lemma 3.4.

The next lemma will be used in the proof of Proposition 3.12 to show that L contains $\mathrm{SL}_n(q)$, except possibly when $n = 2$ or $G = \mathrm{PSL}_5(2)$.

Lemma 3.13. *Let $n \geqslant 3$, let p be a prime, let $q = p^e$ for some $e \geqslant 1$, and let $G \leqslant \Gamma\mathrm{L}_n(q)$ be a group acting on $V = \mathbb{F}_q^n$. Suppose that G acts transitively on $V\backslash\{0\}$ and that for all $v \in V$, the action induced by $G_{\langle v\rangle_q}$ on $V/\langle v\rangle_q$ contains $\mathrm{GL}_{n-1}(q)$. Then either $G \geqslant \mathrm{SL}_n(q)$, or we have $q = 2$, $n = 4$ and $G = \mathrm{Alt}(7)$.*

Proof. Since G acts transitively on $V\backslash\{0\}$, the associated affine group $V \rtimes G$ is 2-transitive, and we can appeal to Lemma 2.9. We may also assume for a contradiction that G does not contain $\mathrm{SL}_n(q)$. Let H be the action induced by $G_{\langle v\rangle_q}$ on $V/\langle v\rangle_q$.

Let us first deal with the case $n = 3$. We see that up to conjugacy, every transitive subgroup of $\mathrm{GL}_3(q)$ that does not contain $\mathrm{SL}_3(q)$ is contained in $\Gamma\mathrm{L}_1(q^3)$. However, we see that in order to have $H \geqslant \mathrm{GL}_2(q)$, we would still need $|G : G_{\langle v\rangle_q}| = q^2 + q + 1$ and $|G_{\langle v\rangle_q}|$ a multiple of $(q^2 - 1)(q^2 - q)$, and then $|G|$ is too large to be a subgroup of $\Gamma\mathrm{L}_1(q^3)$. So from now on we may assume $n \geqslant 4$. In particular, this means G must involve the nonabelian simple group $\mathrm{PSL}_3(q)$ as a quotient of a subgroup.

We next consider the possibility that G can be interpreted as a semilinear group of smaller dimension over a larger field, say $G \leqslant \Gamma\mathrm{L}_m(q^d)$ where $dm = n$ and $d > 1$, and we identify V with $W = \mathbb{F}_{q^d}^m$ as an additive group. In this case $G_{\langle v\rangle_q}$ also stabilizes the $\mathbb{F}_{q^d}$-linear span $\langle v\rangle_{q^d}$ of v. Since G acts irreducibly on $V/\langle v\rangle_q$ as an $\mathbb{F}_q$-vector space, we deduce that $\langle v\rangle_{q^d} = V$, that is,

$d = n$ and $m = 1$. In other words, W is a finite field and each element of G acts as a combination of a multiplication and a field automorphism. But then G is soluble, a contradiction.

Next, consider the case that $n = 2m$ and $p = 2$, and $\mathrm{Sp}_{2m}(q) \trianglelefteq G$. Then $m \geqslant 2$ and $G/\mathrm{Sp}_{2m}(q)$ is soluble, and we would need $\mathrm{Sp}_{2m}(q)_{\langle v \rangle_q}$ to have a quotient $\mathrm{SL}_{n-1}(q) \leqslant Q \leqslant \Gamma\mathrm{L}_{n-1}(q)$. For $m \geqslant 3$ this is easily ruled out by considering the powers of p dividing $|\mathrm{Sp}_{2m}(q)|$ and $|\mathrm{SL}_{2m-1}(q)|$, so we can take $m = 2$. In that case the p'-part of the order of $\mathrm{Sp}_{2m}(q)_{\langle v \rangle_q}$ is

$$\frac{(q^2-1)(q^4-1)}{|P_3(q)|} = (q^2-1)(q-1),$$

whereas the p'-part of $|\mathrm{SL}_3(q)|$ is $(q^2-1)(q^3-1)$, so this case is also ruled out.

Next consider the case that $n = 6$ and $\mathrm{G}_2(q) \trianglelefteq G$. Similar to the last paragraph, we would need $\mathrm{G}_2(q)_{\langle v \rangle_q}$ to have a quotient $\mathrm{SL}_5(q) \leqslant Q \leqslant \Gamma\mathrm{L}_5(q)$, which is easily ruled out by considering the powers of p dividing $|\mathrm{G}_2(q)|$ and $|\mathrm{SL}_5(q)|$.

For $G = \mathrm{Alt}(7)$, $n = 4$, $q = 2$, we have $G_{\langle v \rangle_q} = \mathrm{GL}_3(2)$ acting faithfully on $V/\langle v \rangle_q$.

Finally, suppose G is one the remaining exceptional 2-transitive affine groups of dimension n, where $n \in \{4, 6\}$. Then there is only one insoluble composition factor S of G and it is small: we have $S \leqslant \mathrm{Alt}(6)$ or $S = \mathrm{PSL}_2(13)$. This leaves $\mathrm{Alt}(5)$, $\mathrm{Alt}(6)$ and $\mathrm{PSL}_2(13)$ as the only nonabelian simple groups that can occur as a quotient of a subgroup of G. One sees that this list excludes $\mathrm{PSL}_3(q)$, so we have no further examples. $\square$

Proof of Proposition 3.12. By Lemma 2.6, G has 2-by-block-transitive action on G/L with block stabilizer $G(\alpha_0)$ if and only if (4) is satisfied. We observe that (4) implies (5).

Since the linear part of G acts 2-transitively on the lines in V, we observe that $e_{G(\alpha_0,\alpha_1)} = e_G$. We also deduce from (5) that $G(\alpha_0, \alpha_1) = L(\alpha_1)G_{\mathrm{GL}}(\alpha_0, \alpha_1)$, hence $e_{L(\alpha_1)} = e_{G(\alpha_0,\alpha_1)}$. Let e_p be the largest power of p dividing e_G, and note that $e_p < q$.

Since G has 2-by-block-transitive action on G/L, with block stabilizer $G(\alpha_0)$, we see that L acts transitively on the nontrivial cosets of $G(\alpha_0)$, in other words L acts transitively on $P_n(q)\backslash\{\alpha_0\}$. In particular, given $v, v' \in V\backslash\alpha_0$, there is $g \in L$ such that $gv \in \langle v' \rangle_q$, and then since L contains the scalar matrices, in fact we can ensure $gv = v'$. Thus if we write $\theta : G(\alpha_0) \to \Gamma\mathrm{L}_n(q)$ for the action of $G(\alpha_0)$ on V/α_0, we see that $A := \theta(L)$ is transitive on nonzero vectors.

Write $\beta = \langle v_0, v_1 \rangle_q$ and let $\theta_{V/\beta}$ be the action of $G(\beta)$ on V/β. Since (5) is satisfied and s acts trivially on V/β, and since G contains $Z\mathrm{SL}_{n+1}(q)$, we see that

$$\theta_{V/\beta}(L(\alpha_1)) = \theta_{V/\beta}(G(\alpha_0, \alpha_1)) \geqslant \mathrm{GL}(V/\beta).$$

Given Lemma 3.13, we are therefore in one of the following situations:

(A) $A \geqslant \mathrm{SL}_n(q)$;

(B) $n = 2$ and $A \ngeqslant \mathrm{SL}_n(q)$;

(C) $n = 4$, $q = 2$ and $A = \mathrm{Alt}(7)$.

Observe that W can be regarded as a dual $\mathbb{F}_q$-vector space to V/α_0, interpreting the action of W on V as an inner product as follows: given $v \in V$ and $w \in W$, then $w(v) - v = \lambda_{w,v} v_0$, where the coefficient $\lambda_{w,v} \in \mathbb{F}_q$ only depends on v modulo α_0. We can then define $\langle w, v + \alpha_0 \rangle := \lambda_{w,v}$. In particular, we can identify W with the dual space of V/α_0, and then the conjugation action of $l \in L_{\mathrm{GL}}$ on W is given by $\gamma(l)\rho(\theta(l))$, where ρ is the dual (in other words, inverse transpose) representation of the action of A_{GL} on V/α_0, and $\gamma(l)$ is multiplication by the scalar induced by l on α_0. (In particular, note that $\gamma(l)\rho(\theta(l))$ is trivial if and only if l acts on both V/α_0 and α_0 as multiplication by the same scalar, which happens if and only if $l \in WZ$.) In case (A), assuming $(n, q) \notin \{(2, 2), (2, 3)\}$, we see that the conjugation action of L_{GL} on W contains the perfect self-dual group $\mathrm{SL}(W)$, so the action is transitive on $W\backslash\{1\}$.

Notice that (C) implies (c) by Lemma 3.4. If in addition, we have $L \geqslant W$, then (A) implies (a) and (B) implies (b). So all that remains is to prove $L \geqslant W$ in cases (A) and (B). We finish the proof with a series of claims.

Claim 1: Suppose $n = 2$; then $L \geqslant W$.

Assume for a contradiction that $L \cap W = \{1\}$. Then we see that the largest power of p dividing $|L|$ is at most qe_p, whereas $|G(\alpha_0)|$ is a multiple of $q^3 e_p$. In this context the equation $G(\alpha_0) = sL(\alpha_1)sL$ can only be satisfied if $q^3 e_p \leqslant q^2 e_p^2$, in other words, $q \leqslant e_p$, which is impossible. Thus $L \cap W \neq \{1\}$. Now consider W as the dual $\mathbb{F}_q$-vector space to V/α_0, and let Y represent a line (by which we mean a 1-dimensional subspace) in W. Since L acts transitively on V/α_0, we see that L acts transitively on codimension 1 subspaces, in other words, lines, in W. Thus the intersection $L \cap Y$ has some order $p^{e'}$, where $1 \leqslant e' \leqslant e$, such that e' does not depend on the choice of Y. Write $e'' = e - e'$ and note that $p^{e''} < q$. As there are $q+1$ lines, and distinct lines intersect only at the origin, we have

$$|L \cap W| = (q+1)(p^{e'} - 1) + 1 = qp^{e'} + p^{e'} - q = p^{e'}(q + 1 - p^{e''}).$$

At the same time, $L \cap W$ is a subgroup of W, so $|L \cap W|$ is a power of p, and hence $(q + 1 - p^{e''})$ is a power of p. This can only happen if $p^{e''} = 1$, in other words, $L \geqslant Y$. Then since L is transitive on lines, in fact $L \geqslant W$ as claimed.

Claim 2: Suppose $n = 3$ and $q \in \{2, 3\}$; then $L \geqslant W$.

If $q = 2$ then $|G \backslash G(\alpha_0)| = 2^7 \cdot 3 \cdot 7^2$, so in order to have $G \backslash G(\alpha_0) = LgL$ for some $g \in G$, as in Lemma 2.1, the order of L must be a multiple of $2^4 \cdot 3 \cdot 7 = 336$. A calculation shows that $G(\alpha_0)$ has no proper subgroups of suitable order. If $q = 3$ then G is $\mathrm{PSL}_4(3)$ or $\mathrm{PGL}_4(3)$, and $|G \backslash G(\alpha_0)|$ is a multiple of $2^4 \cdot 3^7 \cdot 13^2$, so $|L|$ must be a multiple of $2^2 \cdot 3^4 \cdot 13 = 4212$. A calculation of maximal subgroups reveals that there is only one case where $G(\alpha_0)$ has a proper subgroup of suitable order, namely when $G = \mathrm{PGL}_4(3)$ and $L = G(\alpha_0) \cap \mathrm{PSL}_4(3)$. However, the latter case is clearly ruled out by Corollary 2.4, proving the claim.

Claim 3: Suppose (A) holds and $\mathrm{SL}_{n-1}(q)$ is perfect; then $L \geqslant W$.

Since L acts transitively on $W \backslash \{1\}$, we may suppose for a contradiction that $L \cap W = \{1\}$.

Recall that $\beta = \langle v_0, v_1 \rangle_q$ and $\theta_{V/\beta}$ is the action of $G(\beta)$ on V/β; let θ_β be the action of $G(\beta)$ on β. Let $G^* = G(\alpha_0, \beta)$ and $L^* = L \cap G^*$. Since L acts transitively on $P_n(q) \backslash \{\alpha_0\}$, we see that $L^* = L(\beta)$ acts transitively on the set X of lines in β other than α_0; note that $|X| = q$. Moreover, since L^*_{GL} is normal in L^* of index dividing e_G, we see that X is partitioned into L^*_{GL}-orbits of equal size, with at most e_p orbits in total. Thus the index $|L^*_{\mathrm{GL}} : L_{\mathrm{GL}}(\alpha_1)|$ is a multiple of $q' := q/e_p$. Since $e_p < q$, we see that q' is a positive power of p.

As noted earlier, we have $\theta_{V/\beta}(L(\alpha_1)) \geqslant \mathrm{GL}(V/\beta)$. Since $\mathrm{SL}(V/\beta)$ is perfect, it follows that $\theta_{V/\beta}(L_{\mathrm{GL}}(\alpha_1)) \geqslant \mathrm{SL}(V/\beta)$. At the same time, we have

$$\theta_{V/\beta}(L_{\mathrm{GL}}(\alpha_1)) \leqslant \theta_{V/\beta}(L^*_{\mathrm{GL}}) \leqslant \mathrm{GL}(V/\beta).$$

Thus the index of $\theta_{V/\beta}(L_{\mathrm{GL}}(\alpha_1))$ in $\theta_{V/\beta}(L^*_{\mathrm{GL}})$ divides $|\mathrm{GL}(V/\beta) : \mathrm{SL}(V/\beta)|$ and hence is coprime to p. In particular, the index

$$|(\ker \theta_{V/\beta} \cap L^*_{\mathrm{GL}}) : (\ker \theta_{V/\beta} \cap L_{\mathrm{GL}}(\alpha_1))|$$

is still a multiple of q'. There is therefore an element $h \in \ker \theta_{V/\beta} \cap L^*_{\mathrm{GL}}$ of p-power order that does not stabilize α_1. We see that h also acts trivially on β/α_0 and α_0.

Let N be the group of elements of G_{GL} that act trivially on V/β, β/α_0 and α_0. Then we see that $N \leqslant M \cap G^*$, so $N \leqslant WL^*$; on the other hand, N is a normal p-subgroup of G^* of order q^{2n-1} that contains W. Thus N is a semidirect product $W \rtimes R$ where $R = L^* \cap N$ is a normal subgroup of L^* of order q^{n-1}. We see that the kernel of the action of R on V/α_0 is contained in W, hence trivial; that is, R acts faithfully on V/α_0. Let $R_2 = R \cap \ker \theta_\beta$. By the previous paragraph, R_2 is properly contained in R; we also see that $|R : R_2| \leqslant q$, so R_2 is nontrivial.

Now consider the image $\theta(N)$ of N in $\mathrm{GL}(V/\alpha_0)$. Since $\theta_{V/\beta}(L^*)$ contains $\mathrm{SL}(V/\beta)$, we see that $\theta(L^*)$ acts transitively by conjugation on $\theta(N)\backslash\{1\}$: the proof is similar to the proof that L acts transitively by conjugation on $W\backslash\{1\}$. But at the same time, $\theta(L^*)$ clearly preserves $\theta(R_2)$ by conjugation, and we have

$$\{1\} < \theta(R_2) < \theta(R) \leqslant \theta(N).$$

This is a contradiction, so we conclude that $L \geqslant W$. This completes the proof of Claim 3 and hence the proposition. □

Given G satisfying Hypothesis 3.9, we now characterize the point stabilizers of the PD actions of G.

Proposition 3.14. *Assume Hypothesis 3.9 and take $M \leqslant L \leqslant G(\alpha_0)$. Then the following are equivalent:*

(i) G has 2-by-block-transitive action on G/L, with block stabilizer $G(\alpha_0)$.

(ii) We have $e_L = e_G$ and $\det(L_{\mathrm{GL}}) = \det(G_{\mathrm{GL}})$.

(iii) We have $e_L = e_G$ and the index $|G(\alpha_0) : L|$ is coprime to $|\mathrm{Pdet}(G_{\mathrm{GL}})|$.

Proof. As in the proof of Proposition 3.12, G has 2-by-block-transitive action on G/L with block stabilizer $G(\alpha_0)$ if and only if the equation (4) is satisfied, and we have $e_{G(\alpha_0,\alpha_1)} = e_G$.

Write $\beta = \langle v_0, v_1\rangle_q$, $G^* = G(\alpha_0, \beta)$ and $L^* = L(\beta)$, and consider the possible equation

$$G^* = L^* sL(\alpha_1)s. \tag{6}$$

We claim that the equations (4), (5) and (6) are equivalent in the present context. We know in general that (4) implies (5). Note that $G(\alpha_0, \alpha_1) \leqslant G^*$. Using the action of $L^* \cap W = W(\beta)$, we see that L^* acts transitively on the 1-dimensional subspaces of β other than α_0, so that $G^* = L^*G(\alpha_0, \alpha_1)$. Thus if $G(\alpha_0, \alpha_1) = L(\alpha_1)sL(\alpha_1)s$, then

$$L^*sL(\alpha_1)s = L^*L(\alpha_1)sL(\alpha_1)s = L^*G(\alpha_0, \alpha_1) = G^*,$$

showing that (5) implies (6). In turn, since the action of L on V/α_0 contains a copy of the special linear group, we see that the L-orbit of β consists of all 1-dimensional subspaces of V/α_0; in other words, $LG^* = G(\alpha_0)$. Thus if (6) holds then

$$LsL(\alpha_1)s = LG^* = G(\alpha_0),$$

showing that (6) implies (4).

We now observe that the subgroup M of L_{GL} acts transitively on 1-dimensional subspaces of V other than α_0, with the result that $e_{L(\alpha_1)} = e_L$. In turn, in order to satisfy (5), we need

$$e_{L(\alpha_1)} = e_{G(\alpha_0,\alpha_1)} = e_G.$$

So from now on we may assume $e_L = e_G$. Note moreover that $M(\alpha_1)$ and $sM(\alpha_1)s$ are both normal in $G(\alpha_0, \alpha_1)$, so if we write $M^* = sM(\alpha_1)sM(\alpha_1)$, then (5) reduces to

$$\frac{G(\alpha_0, \alpha_1)}{M^*} = \frac{L(\alpha_1)sL(\alpha_1)s}{M^*}. \tag{7}$$

We have a homomorphism

$$\phi : G(\alpha_0, \alpha_1) \to \Gamma\mathrm{L}(V/\beta) \times \Gamma\mathrm{L}(\alpha_1) \times \Gamma\mathrm{L}(\alpha_0),$$

given by the action of $G(\alpha_0, \alpha_1)$ on the space $V/\beta \oplus \alpha_1 \oplus \alpha_0$. The assumption $n \geqslant 2$ ensures that V/β is nontrivial. We see that M^* contains the kernel of this action.

In order to make stabilizers of the general linear group more legible, we will write $\Lambda = \mathrm{GL}_{n+1}(q)$. Given $g \in \Lambda(\alpha_0, \alpha_1)$, let $\phi_2(g), \phi_1(g), \phi_0(g)$ be the determinants of g acting on $V/\beta, \alpha_1, \alpha_0$ respectively, and define

$$\delta : \Lambda(\alpha_0, \alpha_1) \to \mathbb{F}_q^*; \quad g \mapsto \phi_2(g)\phi_1(g)\phi_0(g)^{-n}.$$

Then $M(\alpha_1)$ is the kernel of δ, so $\Lambda(\alpha_0, \alpha_1)/M(\alpha_1)$ is cyclic of order $q-1$. Given $g \in \Lambda(\alpha_0, \alpha_1)$, we see that $\delta(g) = \det(g)\phi_0(g)^{-(n+1)}$. If $g \in M(\alpha_1)$ is such that $\phi_i(g) = \mu^{a_i}$ and $\phi_i(sgs) = \mu^{a'_i}$, then we see that $a_2 = na_0 - a_1$; $a'_2 = a_2$; $a'_1 = a_0$; and $a'_0 = a_1$. Hence

$$\delta(sgs) = \mu^{na_0 - a_1}\mu^{a_0}\mu^{-na_1} = \mu^{(n+1)(a_0-a_1)}.$$

Since we can choose a_0 and a_1 freely, we conclude that $M^* = K \ker \delta$ where K is the group of elements of $\Lambda(\alpha_0, \alpha_1)$ of determinant 1. In particular, the quotient map from $\Lambda(\alpha_0, \alpha_1)$ to $\Lambda(\alpha_0, \alpha_1)/M^*$ is equivalent to the map

$$\Lambda(\alpha_0, \alpha_1) \to \mathbb{F}_q^*/\langle \mu^{n+1} \rangle; \quad g \mapsto \det(g)\langle \mu^{n+1} \rangle,$$

which is a restriction of the map Pdet. Note that $\mathrm{Pdet}(s)$ is trivial. We now see that the action of s on $G(\alpha_0, \alpha_1)/M^*$ is trivial, so (7) becomes

$$\frac{G(\alpha_0, \alpha_1)}{M^*} = \frac{L(\alpha_1)M^*}{M^*}, \tag{8}$$

and then since $e_{G(\alpha_0,\alpha_1)} = e_{L(\alpha_1)}$, (8) is equivalent to

$$\frac{G_{\mathrm{GL}}(\alpha_0, \alpha_1)}{M^*} = \frac{L_{\mathrm{GL}}(\alpha_1)M^*}{M^*}. \tag{9}$$

We see that (9) is satisfied if and only if

$$\mathrm{Pdet}(L_{\mathrm{GL}}(\alpha_1)) = \mathrm{Pdet}(G_{\mathrm{GL}}(\alpha_0, \alpha_1)).$$

Since $\mathrm{Pdet}(M)$ is trivial and M acts transitively on lines of V other than α_0, we see that $\mathrm{Pdet}(L_{\mathrm{GL}}(\alpha_1)) = \mathrm{Pdet}(L_{\mathrm{GL}})$. Similarly, $\mathrm{Pdet}(G_{\mathrm{GL}}(\alpha_0, \alpha_1)) = \mathrm{Pdet}(G_{\mathrm{GL}})$. Note also that for any $Z \leqslant H \leqslant G$, then $\det(H_{\mathrm{GL}})$ contains $\langle \mu^{n+1} \rangle$, so $\mathrm{Pdet}(H_{\mathrm{GL}}) = \mathrm{Pdet}(G_{\mathrm{GL}})$ if and only if $\det(H_{\mathrm{GL}}) = \det(G_{\mathrm{GL}})$. Thus (9) is satisfied if and only if $\det(L_{\mathrm{GL}}) = \det(G_{\mathrm{GL}})$. This completes the proof that (i) and (ii) are equivalent.

Finally, we claim that (ii) and (iii) are equivalent; we may suppose $e_G = e_L$, so that $|G_{\mathrm{GL}}(\alpha_0) : L_{\mathrm{GL}}| = |G(\alpha_0) : L|$. We have seen that $\det(L_{\mathrm{GL}}) = \det(G_{\mathrm{GL}})$ if and only if $\mathrm{Pdet}(L_{\mathrm{GL}}) = \mathrm{Pdet}(G_{\mathrm{GL}})$, and moreover $\mathrm{Pdet}(G_{\mathrm{GL}}) = \mathrm{Pdet}(G_{\mathrm{GL}}(\alpha_0))$. Thus (ii) is satisfied if and only if $|G(\alpha_0) : L|$ is coprime to $|\mathrm{Pdet}(G_{\mathrm{GL}})|$, showing that (ii) and (iii) are equivalent. □

We note that if $e_L = e_G$, then to satisfy condition (iii) of Proposition 3.14 it is sufficient but not necessary to have $|G(\alpha_0) : L|$ coprime to $\gcd(n+1, q-1)$. For example, the group $\mathrm{PSL}_3(19)$ has a PD action with block size 3, with point stabilizer $W \rtimes (\mathrm{SL}_2(19) \rtimes C_2)$, that does not extend to an action of $\mathrm{PGL}_3(19)$; since $|\mathrm{Pdet}(\mathrm{GL}_3(19))| = 3$, there is no PD action of $\mathrm{PGL}_3(19)$ with block size 3.

3.4 Some calculations on abelian-by-cyclic groups

To complete the classification of 2-by-block-transitive actions, we need to consider certain subgroups of finite abelian-by-cyclic groups. Specifically, we will be considering the following situation, motivated by Lemma 2.6: G is a finite abelian-by-cyclic group admitting an automorphism s of order 2, and H is a proper subgroup of G such that $G = Hs(H)$.

We first establish some restrictions on quotients of G.

Lemma 3.15. *Let G be a finite group, let α be an automorphism of G and let $H \leqslant G$ be such that $H\alpha(H) = G$.*

(i) Let M be a normal α-invariant subgroup of G such that G/M is cyclic. Then $G = MH$.

(ii) Suppose that $H \cap D$ is α-invariant, where D is the derived group of G. Then $N := H \cap \alpha(H)$ is normal in G and G/N is a group of order $|H : N|^2$.

Proof. (i) In the quotient G/M, we see that the images of H and $\alpha(H)$ have the same order, so $MH = M\alpha(H)$. It is then clear that $G = MH\alpha(H) = MH$.

(ii) Let $R = H \cap D$. Since D is characteristic, we have

$$H \cap D = \alpha(H \cap D) = \alpha(H) \cap D,$$

so in fact $R = N \cap D$. We see that H/R is abelian, so N is normal in H. Similarly, N is normal in $\alpha(H)$. Since $G = H\alpha(H)$ it follows that N is normal in G. The quotient G/N is then the product of two subgroups H/N and $\alpha(H)/N$ with trivial intersection; thus G/N has order $|H : N||\alpha(H) : N| = |H : N|^2$. □

After dividing out by $H \cap sHs$, we will typically be interested in the situation where H is cyclic. Finite groups $G = HK$ such that H and K are cyclic subgroups with trivial intersection were studied by Douglas in a series of articles [6]. We are in effect considering a special case, where in addition some automorphism of G of order 2 swaps H and K. Our focus here is a little different than in [6] however, as we will be classifying factorizations $\overline{G} = Hs(H)$ of quotients $\overline{G}$ of a given group G of a more special form, rather than constructing all finite groups that admit such factorizations.

If $G = A \rtimes H$ for some abelian s-invariant normal subgroup A, then A is also very close to being cyclic.

Lemma 3.16. *Let G be a finite group with an abelian normal subgroup A, such that G/A is cyclic and such that G admits an automorphism s of order 2 that normalizes A. Suppose that there is $H \leqslant G$ such that $H \cap A = \{1\}$ and $G = Hs(H)$. Then A has a cyclic subgroup of index at most 2.*

Proof. We consider G as embedded in a semidirect product $G \rtimes \langle s \rangle$ in the obvious way. By Lemma 3.15(i) we can write $G = A \rtimes H$; note that H is cyclic, say $H = \langle h \rangle$.

We now suppose that (G, A, H, s) is a counterexample with $|G|$ minimal. By Lemma 3.15(ii), the intersection $N = H \cap sHs$ is normal in G; clearly also N is s-invariant, so we can pass from $G \rtimes \langle s \rangle$ to the quotient $G/N \rtimes \langle s \rangle$. By the minimality of G, we deduce that $N = \{1\}$. Then G has order n^2 where $n = |H| = |A|$.

We observe next that A is a p-group for some prime p. Otherwise, we could write $A = A_1 \times A_2$ where A_1 and A_2 are nontrivial and have coprime order. We would then get a smaller counterexample as either $A_1 \rtimes H$ or $A_2 \rtimes H$. So by minimality, A must be a p-group, and hence G is a p-group. Thus $n = p^e$ for some $e \geqslant 1$.

Write $shs = ah$ for some $a \in A$. Since G is finite and $G = HsHs = sHsH$, for all $a' \in A$ there is some $k > 0$ such that $(ah)^k \in a'H$, so $a' = (ah)^k h^{-k}$. We can rearrange $(ah)^k h^{-k}$ as

$$(ah)^k h^{-k} = \prod_{i=0}^{k-1} h^i a h^{-i}. \tag{10}$$

Thus as k ranges over the natural numbers, every element of A must be expressible as a product of conjugates of A as in the right hand side of (10).

Since A has less than n nontrivial elements, the H-conjugacy classes of A all have size at most $q := n/p$. We note also that for all $x \in A$, if m and $qm/2$ are integers then $x^{qm/2} = 1$. If p

is odd this follows from the fact that A is not cyclic; if $p = 2$ and $x^{qm/2} \neq 1$, then x would have order exactly q. But in the latter case, $A = \langle x \rangle \times C_2$, which does not yield a counterexample to the lemma.

Let $Z = \mathrm{Z}(G) \cap A$. Since A is not cyclic, we see from (10) that $a \notin Z$, so Z is a proper subgroup of A. Since $h^q \in \mathrm{Z}(G)$, we can rewrite the product in (10) as follows: writing $k = dq+r$ for $d \geqslant 0$ and $0 \leqslant r < q$, then

$$(ah)^k h^{-k} = b^d \prod_{i=0}^{r-1} h^i a h^{-i}, \tag{11}$$

where $b = (ah)^q h^{-q} = \prod_{i=0}^{q-1} h^i a h^{-i}$; notice that $b \in Z$. If b is trivial, then the right hand side of (10) can only take at most q values, which is a contradiction. So b must be some nontrivial element of Z.

Claim 1: A has exponent dividing 4. If $h^{q/r} a h^{-q/r} \in \langle a \rangle Z$ for some divisor r of q, then $a^r \notin Z$.

We see that G/A_0 is a counterexample to the lemma, where A_0 is the group of fourth powers of A if $p = 2$ and the group of p-th powers of A otherwise. Thus $A_0 = \{1\}$ by minimality of G.

Let r be a divisor of q and let $q' = q/r$. Then we can write b as a product of q' conjugates of b', where

$$b' = \prod_{i=0}^{r-1} h^{q'i} a h^{-q'i}.$$

Suppose $h^{q'} a h^{-q'} = az$ for some $z \in Z$. Then $b' = a^r z^{r(r-1)/2}$. If we also have $a^r \in Z$ then we would have $b = z^{q(r-1)/2} = 1$, a contradiction. So if $h^{q'} a h^{-q'} = az$, then $a^r \notin Z$.

By the minimality of G, the quotient G/Z is not a counterexample to the lemma, so either $p = 2$ or A/Z is cyclic of odd order p. However, since h has p-power order, the latter would imply $hah^{-1} \in aZ$, which has been ruled out. Thus $p = 2$ and A has exponent dividing 4.

Now suppose $aZ \neq a^{-1}Z$ and that $h^{q'} a h^{-q'} = a^{-1}z$. Then r is even and $b' = z^{r(r-1)/2}$, so $b = z^{q(r-1)/2} = 1$, a contradiction. Since the order of a divides 4, we conclude that if $h^{q'} a h^{-q'} \in \langle a \rangle Z$ then $a^r \notin Z$, proving the claim.

Claim 2: A/Z has exponent 4.

Given Claim 1, we may suppose for a contradiction that A/Z has exponent 2. Then A/Z is not cyclic by Claim 1, but also $A/Z \rtimes \langle h \rangle$ satisfies the lemma by the minimality of G, so $A/Z = C_2 \times C_2$ and hence $q \geqslant 4$. We then have $hah^{-1} = ac$ for some $c \in A \backslash Z$ and $hch^{-1} = cz$ for some $z \in Z$, so

$$h^2 a h^{-2} = hach^{-1} = ac^2 z = az.$$

By Claim 1, we see that $a^{q/2} \notin Z$, in particular $a^2 \notin Z$, which contradicts the structure of A/Z. This contradiction completes the proof of the claim.

For the final contradiction, observe that by Claim 2 we have $\Phi(A) \neq \{1\}$; since $G/\Phi(A)$ is not a counterexample to the lemma, it follows that A is generated by at most 2 elements. Given Claim 1, in fact $A = C_4 \times C_4$ and $q = 8$. By Claims 1 and 2, A/Z is not cyclic and has exponent 4, ensuring that $|Z| = 2$. Write $a' = hah^{-1}$. We see from (11) that $a' \notin a\Phi(A)$, so we can write $A = \langle a \rangle \times \langle a' \rangle$. We then see that $b = a^2(a')^2$, and then examining the structure of automorphisms of A of 2-power order, we see that $h^2 a h^{-2}$ takes the form $a^\epsilon b$ for $\epsilon \in \{1, 3\}$. By Claim 1, we have $a^4 \notin Z$, which is impossible. This contradiction completes the proof. □

Example 3.17. Let

$$G = \langle a_1, a_2, h \mid a_1^2,\ a_2^2,\ h^4,\ a_1 a_2 a_1^{-1} a_2^{-1},\ h a_1 h^{-1} a_2^{-1},\ h a_2 h^{-1} a_1^{-1} \rangle,$$

let $A = \langle a_1, a_2 \rangle$ and let $H = \langle h \rangle$. Then G/A is cyclic and G admits an automorphism s of order 2 such that

$$s(a_1) = a_1,\ s(a_2) = a_2,\ s(h) = a_1 h.$$

Using (10) from the proof of Lemma 3.16, one sees that $G = s(H)H = Hs(H)$. However, A is not cyclic. Thus the conclusion of Lemma 3.16, that A has a cyclic subgroup of index at most 2, is sharp.

We now specialize to a case that is relevant for applications to rank 1 groups of Lie type, where we can count the number of conjugacy classes of subgroups $H < G$ of a given index such that $G = sHsH$. To give a succinct expression for the relevant numbers, we define a modified totient function

$$\varphi_k(t) := t \prod_{p \text{ prime},\ p|t,\ p\nmid k} \frac{p-1}{p};$$

this reduces to Euler's totient function when $k = 1$.

Lemma 3.18. *Let $G = \langle x, y \rangle$ be a finite metacyclic group with cyclic normal subgroup $\langle x \rangle$, and suppose G embeds in a semidirect product $G \rtimes \langle s \rangle$. Suppose that s has order 2, normalizes $\langle x \rangle$ and centralizes $G/\langle x \rangle$. Write*

$$yxy^{-1} = x^a;\ sxs = x^{k+1};\ sys = x^l y$$

for $a, k, l \in \mathbb{Z}$ with $a > 0$. Let $k_0 = \gcd(k, l)$; let d_0 be the largest natural number coprime to k_0 that divides $|G : \langle x \rangle|$ and $|\langle x \rangle : \langle y^{|G:\langle x \rangle|} \rangle|$. Write

$$d_0 = 2^{e_0} p_1^{e_1} \dots p_r^{e_r},$$

where $p_1, \dots, p_r$ are distinct odd primes, $e_0 \geqslant 0$ and $e_1, \dots, e_r > 0$. Now set

$$d = 2^{e'_0} p_1^{e'_1} \dots p_r^{e'_r}$$

where

$$e'_0 = \begin{cases} e_0 & \text{if } a \equiv 1 \mod 4 \\ 1 & \text{if } e_0 > 0 \text{ and } a \equiv 3 \mod 4 \\ 0 & \text{otherwise} \end{cases};\quad \forall 1 \leqslant i \leqslant r : e'_i = \begin{cases} e_i & \text{if } a \equiv 1 \mod p_i \\ 0 & \text{otherwise} \end{cases}.$$

Given a natural number n, write $\mathcal{H}_n := \{H < G \mid |G : H| = n, G = HsHs\}$. Then $\mathcal{H}_n$ is a union of G-conjugacy classes, which is nonempty if and only if n divides d. Writing $[\mathcal{H}_n]$ for the set of G-conjugacy classes in $\mathcal{H}_n$, if $\mathcal{H}_n$ is nonempty we have

$$|\mathcal{H}_n| = \varphi_k(n);\quad |[\mathcal{H}_n]| = \varphi_k(\gcd(a-1, n)).$$

Proof. Fix a natural number n. Given $m \in \mathbb{Z}$, write $z_m = x^m y$ and $b_m = km + l$. Let $\alpha(0) = 0$ and thereafter $\alpha(t+1) = a\alpha(t) + 1$, so $\alpha(t) = \sum_{i=1}^{t} a^{i-1}$.

We see that $z_m x z_m^{-1} = x^a$ and that

$$sz_m s = x^{(k+1)m+l} y = x^{b_m} z_m.$$

We now observe that $G = \langle x \rangle \langle z_m \rangle$ and

$$\forall t > 0 : (sz_m s)^t = x^{\alpha(t) b_m} z_m^t. \tag{12}$$

Note that $\alpha(t) \mod n$ is eventually periodic, with some period $n_\alpha \leqslant n$; indeed, the period is the least n' such that $\alpha(t) \equiv \alpha(t + n') \mod n$ for some $t \geqslant 0$. In particular, we see that $n_\alpha = n$ if and only if the following holds: the values $\alpha(0), \dots, \alpha(n-1)$ form a complete set of congruence classes modulo n, and then $\alpha(n) \equiv \alpha(0) \equiv 0 \mod n$. Define U_n to be the set of integers u such that $u \equiv 1 \mod q$ for all q dividing n such that q is a prime or $q = 4$. By the Hull–Dobell Theorem (see [11, §3.2.1.2 Theorem A]), we have $n_\alpha = n$ if and only if $a \in U_n$.

Write $N = \langle x^n, y^n \rangle$; clearly N is normalized by y. If $n_\alpha = n$, then $\alpha(n)$ is a multiple of n. We then have

$$xy^n x^{-1} = (x^{1-a}y)^n = x^{(1-a)\alpha(n)} y^n,$$

so x normalizes N and hence $N \trianglelefteq G$.

We note the following conditions on m, n and a subgroup H of G for future reference.

(a) b_m is a unit modulo n;

(b) $n_\alpha = n$;

(c) $|G : H| = n$;

(d) $H = \langle x^n \rangle \langle z_m \rangle$.

Given $H \in \mathcal{H}_n$, then H must satisfy (c). Since s centralizes $G/\langle x \rangle$, in order to have $G = HsHs$ we must have $G = \langle x \rangle H$, and hence H takes the form $(H \cap \langle x \rangle)\langle z_m \rangle$ for some $m \in \mathbb{Z}$; moreover, $|G : H| = |\langle x \rangle : \langle x \rangle \cap H|$, so $H \cap \langle x \rangle = \langle x^n \rangle$. Thus H also satisfies (d) for some $m \in \mathbb{Z}$. On the other hand, given a subgroup H of G satisfying (c) and (d), then using (12), we see that

$$sHsH = \bigcup_{t>0} \langle x^n \rangle x^{\alpha(t)b_m} \langle z_m \rangle = \bigcup_{t>0} x^{\alpha(t)b_m} H; \tag{13}$$

thus $G = sHsH$, or equivalently $G = HsHs$, if and only if the parameters m and n satisfy (a) and (b).

Suppose $H \leqslant G$ is such that conditions (a)–(d) are satisfied. Then $N \trianglelefteq G$ and

$$z_m^n = (x^m y)^n = x^{\alpha(n)m} y^n \in \langle x^n \rangle y^n,$$

from which it follows that $N = \langle x^n, z_m^n \rangle$; in particular, $N \leqslant H$. Similarly $N \leqslant sHs$, so $N \leqslant H \cap sHs$. The quotient G/N then satisfies $G/N = \langle \overline{x} \rangle \langle \overline{y} \rangle$ where $\overline{x} = xN$ and $\overline{y} = yN$ both have order dividing n. Since $|G : H \cap sHs| = n^2$, in fact $N = H \cap sHs$ and $|G : N| = n^2$, so G/N splits as $\langle \overline{x} \rangle \rtimes \langle \overline{y} \rangle$ with $|\langle \overline{x} \rangle| = |\langle \overline{y} \rangle| = n$. In order for this to occur, we see that n must divide $|G : \langle x \rangle|$ and $|\langle x \rangle : \langle y^{|G:\langle x \rangle|} \rangle|$.

The conclusions we have so far put the following restriction on n in the case that $\mathcal{H}_n$ is nonempty. In order for some $m \in \mathbb{Z}$ to satisfy (a), we need n to be coprime to k_0. The semidirect decomposition of G/N from the last paragraph then ensures that n divides d_0. To additionally satisfy (b), we reduce to the case that n divides d.

For the rest of the proof we suppose that n is a divisor of d; in particular, $n = n_\alpha$, so $N \trianglelefteq G$. Given that every $H \in \mathcal{H}_n$ satisfies condition (d), we see that $\mathcal{H}_n \subseteq \{H_m \mid m \in \mathbb{Z}\}$ where $H_m := N\langle z_m \rangle$, so it is enough to consider subgroups H_m of this form. Given $m \in \mathbb{Z}$, for $0 < k < n$ we see that z_m^k is not in the normal subgroup $N\langle x \rangle$ of G; however, $z_m^n = x^{\alpha(n)m} y^n \in N$, so $|H_m : N| = n$ and hence $|G : H_m| = n$. We also see that $H_m \cap \langle x \rangle = \langle x^n \rangle$. If $m \equiv m' \mod n$ then clearly $H_m = H_{m'}$; conversely, if $H_m = H_{m'}$ then

$$z_{m'} z_m^{-1} = x^{m'-m} \in (H_m \cap \langle x \rangle) = \langle x^n \rangle,$$

so $m \equiv m' \mod n$. We have $G = H_m s H_m s$ if and only if $b_m = km + l$ is coprime to n. Our hypotheses ensure that k and l are coprime modulo n, so there exists m_0 such that $u_0 := km_0 + l$ is a unit modulo n (for instance by Dirichlet's theorem on primes in arithmetic progressions). The set of possible values of $km + l$ modulo n is then provided by the set $\{u_0 + kt \mid t \in \mathbb{Z}\}$. Write

n_0 for the largest factor of n coprime to k; the proportion of values of t (modulo n) for which $u_0 + kt$ is a unit modulo n is

$$\prod_{p \mid n_0} \frac{p-1}{p} = \frac{\varphi_k(n)}{n}.$$

Thus $|\mathcal{H}_n| = \varphi_k(n)$.

Now suppose $H \in \mathcal{H}_n$ and consider the conjugates of H in G. Since $G = \langle x \rangle H$, it is enough to consider conjugation by $\langle x \rangle$. Here we find that

$$xH_m x^{-1} = N\langle x^{m+1} y x^{-1} \rangle = N\langle x^{m+(1-a)} y \rangle = H_{m+(1-a)},$$

and by our assumption on n, the number $(1-a)$ is divisible by every prime that divides n. In particular, $km + l$ is a unit modulo n if and only if $k(m + (1-a)) + l$ is a unit modulo n, so $\mathcal{H}_n$ is invariant under conjugation by $\langle x \rangle$; hence $\mathcal{H}_n$ is a union of conjugacy classes of G. Write $a' = \gcd(a-1, n)$; then n and a' have the same prime divisors, so $\varphi_k(a')/a' = \varphi_k(n)/n$. The orbits of the action of $\langle x \rangle$ on $\mathcal{H}_n$ have size n/a', so each $H \in \mathcal{H}_n$ has normalizer in G of index n/a'. Thus the number of G-conjugacy classes in $\mathcal{H}_n$ is

$$|[\mathcal{H}_n]| = \frac{a'|\mathcal{H}_n|}{n} = \frac{a'\varphi_k(n)}{n} = \varphi_k(a'). \qquad \square$$

Remark 3.19. In the situation of Lemma 3.18, with $n > 1$, then the set $\mathcal{H}_n$ admits an action of s that does not preserve any G-conjugacy class, and hence $|[\mathcal{H}_n]|$ must be even. We can check this from the formula for $|[\mathcal{H}_n]|$ in the case that $\mathcal{H}_n$ is nonempty, as follows. Write $a' = \gcd(a-1, n)$. If n is even then k is even (since $k+1$ is a unit modulo n) and a' is even; thus $\varphi_k(a')$ is even. If some odd prime p divides n but not k, then $p-1$ divides $\varphi_k(a')$, so again $\varphi_k(a')$ is even. Thus we may assume that n is odd and that every prime dividing n also divides k, implying in particular that $k+2$ is coprime to n. However, given that $s^2 = 1$, we see that

$$y = s(sys)s = x^{(k+2)l} y,$$

so $(k+2)l$ is a multiple of n. Then l is a multiple of n, so every prime dividing n divides $\gcd(k, l)$, which is impossible for $n > 1$.

3.5 Groups of rank 1 Lie type

We will say a 2-transitive permutation group G is of **small rank Lie type** if its socle is a nonabelian simple group of one of the following forms, in its natural 2-transitive action:

$$\mathrm{PSL}_2(q)\ (q \geqslant 4),\ \mathrm{PSU}_3(q)\ (q \geqslant 3),\ {}^2\mathrm{B}_2(2^e)\ (e \geqslant 3 \text{ odd}),\ {}^2\mathrm{G}_2(3^e)\ (e \geqslant 3 \text{ odd}),\ \mathrm{PSL}_3(q).$$

The first four types are of Lie rank 1, while $\mathrm{PSL}_3(q)$ has Lie rank 2. (The group ${}^2\mathrm{G}_2(3) \cong \mathrm{P\Gamma L}_2(8)$ in its 2-transitive action on 28 points is also of rank 1 Lie type, however we exclude it from this discussion as it has already been dealt with in Lemma 3.2.) To clarify, the underlying field $\mathbb{F}$ of the group is $\mathbb{F}_{q^2}$ in the unitary case and $\mathbb{F}_q$ otherwise, where $q = p^e$ is a power of the prime p; we take the 2-transitive action of $\mathrm{PSL}_{n+1}(q)$ to be on the set $\Omega_0 = P_n(q)$ of lines in $V = \mathbb{F}_q^{n+1}$, which extends to an action of $\mathrm{P\Gamma L}_{n+1}(q)$.

We set another hypothesis to establish some notation. Note that this hypothesis is compatible with Hypothesis 3.9 in the case of socle $\mathrm{PSL}_3(q)$.

Hypothesis 3.20. We suppose that G is a group acting on the set Ω_0. Write Z for the kernel of the action of G on Ω_0. We suppose one of the following holds:

(i) $Z = \{1\}$ and G has socle S of rank 1 Lie type, with Ω_0 being the natural 2-transitive S-set;

(ii) We have $G \leqslant \Gamma\mathrm{L}_3(q)$ and Ω_0 is the set of lines in $V = \mathbb{F}_q^3$; Z is the group of scalar matrices in $\mathrm{GL}_3(q)$; and $G \geqslant S$ where $S = Z\mathrm{SL}_3(q)$.

Enlarge G to the 2-transitive group $\overline{G}$ and set the parameter t_0, as follows:

(a) If $S = \mathrm{PSL}_2(q)$ and q is odd, set $t_0 = 2$ and $\overline{G} = \langle \mathrm{PGL}_2(q), G\rangle$.

(b) If $S = \mathrm{PSU}_3(q)$ and $q+1$ is a multiple of 3, set $t_0 = 3$ and $\overline{G} = \langle \mathrm{PGU}_3(q), G\rangle$.

(c) If $S = Z\mathrm{SL}_3(q)$ and $q-1$ is a multiple of 3, set $t_0 = 3$ and $\overline{G} = \langle \mathrm{GL}_3(q), G\rangle$.

(d) Otherwise, set $t_0 = 1$ and $\overline{G} = G$.

Let $\mathbb{F}$ be the underlying field of S. If $S \in \{\mathrm{PSL}_2(q), \mathrm{PSU}_3(q)\}$, let $\tilde{G}$ be the central extension of $\overline{G}$ obtained by lifting S to the corresponding special linear/unitary group, otherwise let $\tilde{G} = \overline{G}$. Then $\tilde{G}$ has a standard action by semilinear maps on $V = \mathbb{F}^{r(S)}$ where

$$r(\mathrm{PSL}_{n+1}(q)) = n+1,\ r(\mathrm{PSU}_3(q)) = 3,\ r({}^2\mathrm{B}_2(q)) = 4,\ r({}^2\mathrm{G}_2(q)) = 7.$$

We let $\tilde{G}_{\mathrm{GL}}$ be the subgroup of $\mathbb{F}$-linear elements of $\tilde{G}$, let $\overline{G}_{\mathrm{GL}}$ be the image of $\tilde{G}_{\mathrm{GL}}$ in $\overline{G}$ and let $G_{\mathrm{GL}} = \overline{G}_{\mathrm{GL}} \cap G$. Let $e_G = |G : G_{\mathrm{GL}}|$ and let $f_G = 2e/e_G$ in the unitary case and $f_G = e/e_G$ otherwise.

If G is of rank 1 Lie type, we take $x_0 \in \overline{G}_{\mathrm{GL}}$ to be a diagonal element generating a maximal torus. We also have $\phi^{f_G} \in \tilde{G}$ where ϕ is the standard Frobenius map on V; we write y_0 for the image of ϕ^{f_G} in $\overline{G}$ and note that y_0 normalizes $\langle x_0\rangle$.[1] We take points $\omega, \omega' \in \Omega_0$ such that $\overline{G}(\omega, \omega') = \langle x_0\rangle \rtimes \langle y_0\rangle$ (see [5, §7.7]), and then take $s \in S$ of order 2 such that s swaps ω and ω' and commutes with y_0, and such that $sx_0s = x_0^{k+1}$, where if $S = \mathrm{PSU}_3(q)$ then $k = -(q+1)$, otherwise $k = -2$.

For $S = Z\mathrm{SL}_3(q)$, we choose ω and ω' to be the standard lines α_0 and α_1 respectively in V; we take s as in Hypothesis 3.9. We consider the decomposition

$$\overline{G}_{\mathrm{GL}}(\alpha_0) = W \rtimes Z\Lambda$$

where $\Lambda \cong \mathrm{GL}_2(q)$ fixes v_0 and acts in the natural manner on $\langle v_1, v_2\rangle_q$, take a Singer cycle $\hat{x}_0 \in \Lambda$, and then set $x_0 = \hat{x}_0^{q+1}$. Notice that x_0 acts as a scalar on $\langle v_1, v_2\rangle_q$, so $x_0\alpha_1 = \alpha_1$; on the other hand, $W\langle \hat{x}_0\rangle$ acts transitively on $P_2(q)\backslash\{\alpha_0\}$. Both x_0 and sx_0s are diagonal matrices in the standard basis, so they commute. Writing $R = WZ\langle \hat{x}_0\rangle$, we see that $\mathrm{N}_{\overline{G}(\alpha_0)}(R)$ can be put in a form

$$\mathrm{N}_{\overline{G}(\alpha_0)}(R) = WZ \rtimes \langle \hat{x}_0, y_0\rangle,$$

where we can regard $\langle \hat{x}_0\rangle$ as a copy of $\mathrm{GL}_1(q^2)$, and where y_0 has order $2e_G$, with

$$y_0\hat{x}_0y_0^{-1} = \hat{x}_0^{p^{f_G}}.$$

Since R acts transitively on $P_2(q)\backslash\{\alpha_0\}$, we can choose $y_0 \in \overline{G}(\alpha_0, \alpha_1)$. Let Z^* be as defined in Lemma 3.11 and let $x_* = sx_0sx_0^{-1}$. One finds that $sy_0sy_0^{-1} \in Z^*\langle x_*\rangle$; for a later argument we will want to ensure that in fact $sy_0sy_0^{-1} \in Z^*\langle x_*^3\rangle$, which we can achieve by replacing y_0 with $x_*^a y_0$ for a suitable $a \in \mathbb{Z}$. After multiplying by an element of Z we can ensure that y_0 fixes v_0. The element $y_0^{e_G}$ then fixes v_0 and belongs to $\mathrm{GL}_3(q)$, so $y_0^{e_G} \in \Lambda$.

[1] There are some complications in attaching a field automorphism to a unitary group, but there is no ambiguity with socle $\mathrm{PSU}_3(q)$, see [3].

In all cases,

$$\overline{G} = (S\langle x_0\rangle)\langle y_0\rangle,$$

where $S\langle x_0\rangle = \overline{G}_{\mathrm{GL}}$, and we have $|(S\langle x_0\rangle) \cap \langle y_0\rangle| = 2$ if $S = Z\mathrm{SL}_3(q)$ and $|(S\langle x_0\rangle) \cap \langle y_0\rangle| = 1$ otherwise.

Write $x = x_0^{t_G}$ for the smallest positive power of x_0 contained in G (so $t_G \in \{1, t_0\}$). For $S = Z\mathrm{SL}_3(q)$, we also have the smallest power $\hat{x} := \hat{x}_0^{t_G}$ of $\hat{x}_0$ contained in G, so $\langle x\rangle = \langle \hat{x}^{q+1}\rangle$. Similarly, the element $y_0 \in \overline{G}$ is not always an element of G; the most we can ensure is that there is an element $y := x_0^{r_G} y_0$ of G, where $0 \leqslant r_G < t_G$. (The details of how we have chosen y_0 will become relevant when $t_G > 1$, as in this case we will find some differences in 2-by-block-transitive actions between the case $r_G = 0$ and the case $r_G \neq 0$.)

With the almost simple 2-transitive groups of rank 1 Lie type, the stabilizer of a pair of points in Ω_0 is cyclic or metacyclic. We can thus apply the results of the previous subsection.

Proposition 3.21. *Let G be a group satisfying Hypothesis 3.20, of rank 1 Lie type acting 2-transitively on the set Ω_0. Write $\mathcal{H}_n$ for the class of subgroups $L \leqslant G(\omega)$ such that $|G(\omega) : L| = n$ and G has 2-by-block-transitive action on G/L.*

(i) We can write $G(\omega) = P \rtimes \langle x, y\rangle$, where P is a p-group acting regularly on $\Omega_0 \backslash \{\omega\}$.

(ii) Let $L \in \mathcal{H}_n$ for some $n \geqslant 1$. Then $L = PN\langle z\rangle$, where

$$z \in \langle x\rangle y;\ N = \langle x^n, y^n\rangle \trianglelefteq \langle x, y\rangle.$$

The quotient $G(\omega)/PN$ is of the form $\langle \overline{x}\rangle \rtimes \langle \overline{z}\rangle$ where $\langle \overline{x}\rangle$ and $\langle \overline{z}\rangle$ both have order n.

(iii) Set $o_G = |\langle x\rangle : \langle y^{e_G}\rangle|$ and write $\gcd(e_G, o_G) = 2^{e_0} p_1^{e_1} \dots p_r^{e_r}$, where $p_1, \dots, p_r$ are distinct odd primes. For each of the odd primes p_i define a condition

(U_i) *$S = \mathrm{PSU}_3(q)$; p_i divides $q+1$; for $p_i = 3$ we also require that 9 divides $q+1$ or $r_G = 0$.*

We set

$$e_0' = \begin{cases} e_0 & \text{if } t_G = 2,\ e_G \text{ is even},\ r_G = 1,\ p^{f_G} \equiv 1 \mod 4 \\ 1 & \text{if } t_G = 2,\ e_G \text{ is even},\ r_G = 1,\ p^{f_G} \equiv 3 \mod 4 \\ 0 & \text{otherwise} \end{cases}.$$

For the exponents of the odd primes p_i, we set

$$e_i' = \begin{cases} e_i & \text{if } p^{f_G} \equiv 1 \mod p_i \text{ and } (\mathrm{U}_i) \text{ is false} \\ 0 & \text{otherwise} \end{cases}.$$

Then the values of n for which $\mathcal{H}_n$ is nonempty are the divisors of

$$d_G = 2^{e_0'} p_1^{e_1'} \dots p_r^{e_r'}.$$

(iv) Let $n > 1$ be a divisor of d_G and let h_n be the number of $G(\omega)$-conjugacy classes in $\mathcal{H}_n$. Then

$$h_n = \varphi_{t_G}(\gcd(p^{f_G} - 1, n)).$$

Proof. We have $\overline{G}(\omega) = P \rtimes \overline{G}(\omega, \omega')$, where P is a p-group contained in S that acts regularly on $\Omega \backslash \{\omega\}$ (see [5, §7.7]), and $\overline{G}(\omega, \omega') = \langle x_0\rangle \rtimes \langle y_0\rangle$. The description of $G(\omega)$ in (i) follows easily. Note that $\langle S, x\rangle = G_{\mathrm{GL}}$ has index e_G in G.

Let $L \in \mathcal{H}_n$ for some $n \geqslant 2$. Then $L(\omega')sL(\omega')s = \langle x, y\rangle$ by Lemma 2.6. The group $\langle x, y\rangle$ is metacyclic and the normal subgroup $\langle x\rangle$ is normalized by s. Note that k is a multiple of t, hence a multiple of t_G. We see that

$$sys = sx_0^{r_G}sy_0 = x_0^{kr_G}y = x^{l_G}y \in \langle x\rangle y,$$

where $l_G = kr_G/t_G$; note that l_G can be nonzero only if $t_G > 1$. In particular, s centralizes the quotient $\langle x, y\rangle/\langle x\rangle$. We can thus apply Lemma 3.18 to limit the possibilities for $L(\omega')$.

Since x has order coprime to p, Lemma 3.18 ensures that $L(\omega')$ has index in $\langle x, y\rangle$ coprime to p. We deduce that $|G(\omega) : L|$ is likewise coprime to p, and thus $L = PL(\omega')$. Using Lemma 2.6 and given $L \leqslant G(\omega)$, we now see that G has 2-by-block-transitive action on G/L if and only if $P \leqslant L$ and $G(\omega, \omega') = L(\omega')sL(\omega')s$; thus the groups $L \in \mathcal{H}_n$ are exactly the products PM such that $|\langle x, y\rangle : M| = n$ and $\langle x, y\rangle = MsMs$.

The statement (ii) now follows directly from Lemma 3.18. Note also that $L \cap sLs = N$, and the $G(\omega)$-conjugacy classes of $\mathcal{H}_n$ naturally correspond to conjugacy classes of subgroups M of $\langle x, y\rangle$ of index n satisfying $\langle x, y\rangle = MsMs$.

We have

$$yx_0y^{-1} = x^{a_G}, \text{ where } a_G = p^{f_G}.$$

We also have

$$y^{e_G} = x_0^{r'_G}, \text{ where } r'_G = r_G \sum_{i=0}^{e_G-1} a_G^i.$$

In particular, for $t_G > 1$ the value of r_G is subject to the additional constraint that $y^{e_G} \in \langle x\rangle$, so r'_G will be a multiple of t_G. Set

$$o_G := |\langle x\rangle : \langle y^{e_G}\rangle| = \gcd(|\mathbb{F}^*|/t_G, r'_G/t_G).$$

As in Lemma 3.18 we write $k_0 = \gcd(k, l_G)$. We have the following cases:

(A) If $S = \mathrm{PSL}_2(q)$, $t_G = 2$ and $r_G = 1$, then $k_0 = 1$;

(B) If $S = \mathrm{PSU}_3(q)$, then either $k_0 = q+1$ or $k_0 = (q+1)/3$, with the latter occurring if $t_G = 3$, $r_G > 0$ and $q + 1$ is a multiple of 3;

(C) Otherwise, $k_0 = 2$.

Starting from

$$\gcd(e_G, o_G) = 2^{e_0}p_1^{e_1} \dots p_r^{e_r},$$

we take a divisor $d_G = 2^{e'_0}p_1^{e'_1} \dots p_r^{e'_r}$, in such a way that the divisors of d_G satisfy the conditions set out in Lemma 3.18. Specifically, the exponents e'_i are taken as follows.

For the exponent of 2, in case (B) we see that $e'_0 = 0$ if $p = 2$ (since o_G is odd) and also if $p > 2$ (since k_0 is even). In case (C), k_0 is even, so again we have $e'_0 = 0$. So let us assume we are in case (A). We may also assume that e_G and $q - 1$ are even; in particular, q is an even power of an odd prime, so $q \equiv 1 \mod 4$. Under these assumptions,

$$\gcd(e_G, o_G) = \gcd\left(e_G, \frac{q-1}{2}, \frac{1}{2}\sum_{i=0}^{e_G-1} a_G^i\right).$$

If $a_G \equiv 3 \mod 4$ we see that $\gcd(e_G, o_G)$ is even, and we take $e'_0 = 1$. If instead $a_G \equiv 1 \mod 4$, we take $e'_0 = e_0$, that is, the exponent of the largest power of 2 dividing $\gcd(e_G, o_G)$.

For odd primes, we need to exclude the prime divisors of k_0. In cases (A) and (C) there are no odd prime divisors of k_0. In case (B), the odd prime divisors of k_0 are the same as those of $q + 1$, with the following exception: if $r_G > 0$ (implying that $t_G = 3$) and $q + 1$ is not divisible by 9, then 3 divides $q + 1$ but not k_0. Thus the prime p_i divides k_0 if and only if (U_i) holds.

We now take

$$e_i' = \begin{cases} e_i & \text{if } p^{f_G} \equiv 1 \mod p_i \text{ and } (\mathrm{U}_i) \text{ is false} \\ 0 & \text{otherwise} \end{cases}.$$

Part (iii) now follows from Lemmas 2.6 and 3.18.

It remains to count the conjugacy classes, using the formula from Lemma 3.18. If n is even, then $e_0' > 0$ and we see that we are in case (A), so $k = -2$ and $t_G = 2$. Lemma 3.18 then yields

$$h_n = \varphi_{-2}(\gcd(a_G - 1, n)) = \varphi_{t_G}(\gcd(p^{f_G} - 1, n)).$$

From now on we may assume n is odd. If the socle is $\mathrm{PSU}_3(q)$ then we have $k = -(q+1)$, so

$$h_n = \varphi_{q+1}(\gcd(p^{f_G} - 1, n)).$$

However, we know that n is coprime to all prime divisors of $q+1$, except possibly the prime 3; if 3 divides both $q+1$ and n, then we are in the case where $t_G = 3$. So in fact

$$h_n = \varphi_{t_G}(\gcd(p^{f_G} - 1, n)).$$

In the remaining case,

$$h_n = \varphi_2(\gcd(p^{f_G} - 1, n)) = \varphi(\gcd(p^{f_G} - 1, n)) = \varphi_{t_G}(\gcd(p^{f_G} - 1, n)).$$

This completes the proof of part (iv). □

We deduce that the resulting action is never sharply 2-by-block-transitive.

Corollary 3.22. *Let G and Ω_0 be as in Proposition 3.21. Then the action of G on Ω_0 does not extend to a sharply 2-by-block-transitive action of G.*

Proof. We retain the notation of Proposition 3.21; it is enough to consider the action of G on G/L for $L \in \mathcal{H}_n$. From the structure of the point stabilizer, we see that N is the stabilizer of a distant pair in G/L and N is a subgroup of $\langle x, y\rangle$ of index n^2. So in order to have a sharply 2-by-block-transitive action with block size n, in the notation of Proposition 3.21 we would need

$$|\langle x \rangle| = o_G = e_G = d_G = n.$$

We note that the equation $|\langle x\rangle| = |e_G|$ is rarely satisfied. In the non-unitary case, $|\langle x\rangle| = (p^e - 1)/t_G$ and e_G divides e, so in order to have $|\langle x\rangle| = |e_G|$ we would need

$$p^e \leqslant t_G e + 1.$$

If $t_G = 1$, the above inequality is only satisfied when $p = 2$ and $e = 1$. If $t_G = 2$ then $p \geqslant 3$, and the only solution is $(p, e) = (3, 1)$. So we are left with the groups $\mathrm{PSL}_2(2) \cong \mathrm{Sym}(3)$ and $\mathrm{PSL}_2(3) \cong \mathrm{Alt}(4)$, both of which are soluble.

In the unitary case, $|\langle x\rangle| = (p^{2e} - 1)/t_G$ and e_G divides $2e$, so we would need

$$p^{2e} \leqslant 2t_G e + 1.$$

If $t_G = 1$ there are no solutions. If $t_G = 3$ then $p^e \equiv 2 \mod 3$, and to satisfy the inequality we would need $p = 2$ and $e = 1$. Thus $G = \mathrm{PSU}_3(2) \cong C_3^2 \rtimes Q_8$, acting on 9 points. However, in this case G is again a soluble group. □

Remark 3.23. The three groups $\mathrm{PSL}_2(2), \mathrm{PSL}_2(3), \mathrm{PSU}_3(2)$ appearing in the proof of Corollary 3.22 are all sharply 2-transitive in their natural action, thus sharply 2-by-block-transitive with trivial blocks. These examples are excluded from the context of Proposition 3.21, since we assume G has nonabelian simple socle.

We also rule out the possibility of proper 3-by-block-transitive actions arising from Proposition 3.21. Since the action on blocks must be 3-transitive, we only need to consider the case where G has socle $\mathrm{PSL}_2(q)$ and Ω_0 is the projective line.

Proposition 3.24. *Let $\mathrm{PSL}_2(q) \leqslant G \leqslant \mathrm{P\Gamma L}_2(q)$ be such that the action of G on the projective line Ω_0 is 3-transitive, and suppose Ω_0 extends to a 2-by-block-transitive action on $\Omega = \Omega_0 \times B$, with block size $|B| = n > 1$. Then G has cn^2 equally-sized orbits on $\Omega^{[3]}$, where $c = 2$ if n is even and $c = 1$ otherwise. In particular, G is not 3-by-block-transitive.*

Proof. We retain the notation of Proposition 3.21. We can take a block stabilizer of the form $G([\omega_1]) = P \rtimes \langle x, y\rangle$. The action of $P \rtimes \langle x, y\rangle$ is transitive on the points of $\Omega\backslash[\omega_1]$, with $A = \langle x, y\rangle$ as the stabilizer in $G([\omega_1])$ of a block $[\omega_2]$ and a point stabilizer $H = G([\omega_1], \omega_2)$ of index n in A. Note that A acts transitively on $[\omega_2]$, so H only depends on the choice of representative ω_2 up to conjugation in A. Let Ω_1 be the set of blocks of Ω other than $[\omega_1]$ and $[\omega_2]$; thus $|\Omega_1| = q - 1$. Considering the action of $\langle x_0\rangle$ on Ω_0, we see that action of $\langle x\rangle$ on Ω_1 is free, so it has t_G orbits. Meanwhile, given that $y = x_0^{r_G} y_0$ where y_0 acts on Ω_0 as a field automorphism, we see that if $r_G = 0$, then $\langle y\rangle$ stabilizes some $[\omega_3] \in \Omega_1$, whereas if $r_G = 1$ then y swaps the two orbits of $\langle x\rangle$ on Ω_1. Overall, A must act transitively on Ω_1 for G to be 3-transitive, so either $t_G = 1$ or $t_G = 2$ and $r_G = 1$. We then see that $y_0^{t_G}$ is the smallest power of y_0 contained in G, and we have

$$\langle y_0^{t_G}\rangle = G([\omega_1], [\omega_2], [\omega_3]); \quad K := G([\omega_1], \omega_2, [\omega_3]) = \langle y_0^{t_G}\rangle \cap H.$$

Thus $\langle y_0^{t_G}\rangle \cap H^*$ fixes $[\omega_2]$ pointwise, where H^* is the intersection of all A-conjugates of H. From Proposition 3.21 we see that $H^* \geqslant N := \langle x^n, y^n\rangle$. Thus $\langle y_0^{t_G n'}\rangle$ fixes $[\omega_2]$ pointwise, where n' is the least exponent such that $y_0^{t_G n'} \in N$. If n is odd we see that $n' = n$; if n is even, then $t_G = 2$ and $y_0^2 = x^{2c'} y^2$ for some $c' \in \mathbb{Z}$, so $n' = n/2$. Since G is 3-transitive on Ω_0, we find that for each $\langle y_0^{t_G}\rangle$-invariant block $[\omega']$, there is some $g \in \mathrm{N}_G(\langle y_0^{t_G}\rangle)$ such that $[\omega'] = [g\omega_2]$. Thus $\langle y_0^{t_G n'}\rangle$ fixes pointwise every $\langle y_0^{t_G}\rangle$-invariant block.

The stabilizer of the pair (ω_1, ω_2) is N, so we must count the number of N-orbits on $\Omega\backslash([\omega_1] \cup [\omega_2])$. First consider the action of N on Ω_1: here the orbits are equally-sized since N is normal in A, and the number of orbits is

$$|A : N\langle y_0^{t_G}\rangle| = \frac{n^2}{|N\langle y_0^{t_G}\rangle : N|} = \frac{n^2}{n'} = cn.$$

Then if we consider the N-orbits on $\Omega\backslash([\omega_1] \cup [\omega_2])$ passing through the block $[\omega_3]$, we see that their intersections with $[\omega_3]$ are the orbits of

$$G(\omega_1, \omega_2, [\omega_3]) = \langle y_0^{t_G}\rangle \cap N = \langle y_0^{t_G n'}\rangle.$$

However, we have established that $\langle y_0^{t_G n'}\rangle$ fixes $[\omega_3]$ pointwise, so there are n orbits of N passing through $[\omega_3]$. The same situation will arise for any block $[\omega'] \in \Omega_1$, with some A-conjugate of $\langle y_0^{t_G n'}\rangle$ playing the role of $\langle y_0^{t_G n'}\rangle$. Thus N has cn^2 equally-sized orbits on $\Omega\backslash([\omega_1] \cup [\omega_2])$. Since G is transitive on distant pairs, we deduce that G has cn^2 equally-sized orbits on distant triples. ☐

Example 3.25. Let p be prime and let n be coprime to $2p$, $n > 1$. Let m be the multiplicative order of p modulo n and suppose that m is odd; for example, one can take (p, n, m) to be $(2, 7, 3)$, or for examples where n is not a prime power, take (p, n, m) to be $(2, 161, 33)$ or $(3, 143, 15)$. Write $q = p^{mn}$. Then the field of order q^2 has an automorphism y of order $2n$ given by $\lambda \mapsto \lambda^{p^m}$, which restricts to an automorphism of order n of the field of order q. Note that we have ensured $q \equiv 1 \mod n$; since n is odd it follows that $q + 1$ is a unit modulo n, whereas $q - 1$ is a multiple of n.

Set $G = \mathrm{PGU}_3(q) \rtimes \langle y \rangle$, in its standard 2-transitive action on a set Ω. Then G has a point stabilizer $G(\omega) = P \rtimes G(\omega, \omega')$ where P is a p-group acting regularly on $\Omega \backslash \{\omega\}$ and $G(\omega, \omega') = \langle x, y \rangle$, where x generates a copy of $\mathbb{F}^*_{q^2}$. We also have an involution $s \in G$ such that $s \notin \langle x, y \rangle$, $sxs = x^{-q}$ and $sys = y$. We have ensured that the order of x is divisible by n; let $N = \langle x^n, y^n \rangle$. Write $\overline{a}$ for the image of an element a under the quotient map $\langle x, y, s \rangle \to \langle x, y, s \rangle / N$. Since $p^m \equiv 1 \mod n$, the group $\langle x, y \rangle / N$ is abelian and takes the form of a direct product $\langle \overline{x} \rangle \times \langle \overline{y} \rangle$ with both factors being cyclic of order n. Writing $\overline{z} = \overline{xy}$, then $\overline{z}$ has order n and $\overline{szs} = \overline{x}^{-q}\overline{z}$. We deduce from Proposition 3.21 that $\langle x, y \rangle / N = \langle \overline{z} \rangle \overline{s} \langle \overline{z} \rangle \overline{s}$. We thus obtain a proper 2-by-block-transitive action of G with point stabilizer

$$L = P \rtimes \langle x^n, y^n, xy \rangle,$$

which is normal in $G(\omega)$ of index n.

Similarly, we can take $G = \mathrm{PGL}_2(q) \rtimes \langle y \rangle$, again with the standard 2-transitive action on a set Ω, where now we take y to have order n (in order to make G almost simple). We obtain a proper 2-by-block-transitive action of G in the same manner as the previous paragraph, with only the following minor differences: this time x generates a copy of $\mathbb{F}^*_q$, and the involution s is such that $sxs = x^{-1}$. If $p = 2$ or $p = 3$, the same construction applies to $G = {}^2\mathrm{B}_2(2^{mn}) \rtimes \langle y \rangle$ or $G = {}^2\mathrm{G}_2(3^{mn}) \rtimes \langle y \rangle$ respectively.

In the previous two paragraphs, we have arranged that $t_G = 1$. The number of equivalence classes of 2-by-block-transitive actions of G extending the standard 2-transitive action with block size n is therefore

$$h_n = \varphi(\gcd(p^m - 1, n)).$$

So for instance if $(p, n, m) = (2, 7, 3)$, we have $\gcd(p^m - 1, n) = 7$ and $h_n = 6$, and thus for the field automorphism $y : \lambda \mapsto \lambda^{2^3}$, we obtain 6 permutationally inequivalent 2-by-block-transitive actions for G, where G is one of the groups

$$\mathrm{PGL}_2(2^{21}) \rtimes \langle y \rangle,\ {}^2\mathrm{B}_2(2^{21}) \rtimes \langle y \rangle,\ \mathrm{PGU}_3(2^{21}) \rtimes \langle y \rangle,$$

acting on $7(2^{21} + 1), 7(2^{42} + 1), 7(2^{63} + 1)$ points respectively with blocks of size 7.

The existence of these examples can be contrasted with the fact that the groups $\mathrm{PSU}_3(p^{mn})$, ${}^2\mathrm{B}_2(2^{mn})$ and ${}^2\mathrm{G}_2(3^{mn})$ do not occur as the socle of any imprimitive rank 3 permutation group, see [4, Propositions 4.7, 4.8, 4.9]. Indeed the rank of the imprimitive permutation groups produced in this example (in other words, the number of double cosets of L in G) is $n + 1$, and we needed to assume $n > 1$ and n odd for the construction.

3.6 Quadratic-extended projective planes

Before continuing with the classification of finite block-faithful 2-by-block-transitive actions, we consider a geometric construction that produces both finite examples and infinite examples of sharply 2-by-block transitive groups. The construction is not strictly necessary for the proof of the main theorems, but it provides a better intuition for why case (b) of Theorem 1.3 arises than the direct analysis of subgroup structure. The idea of the construction is due to Hendrik Van Maldeghem (oral communication).

Example 3.26. Let K be a (finite or infinite) field and let $L = K(\alpha)$ be a quadratic extension of K. We form the L-vector space $L^3 = Lv_0 \oplus Lv_1 \oplus Lv_2$; within L^3, we have a K-subspace $K^3 = Kv_0 \oplus Kv_1 \oplus Kv_2$. The group $\mathrm{PGL}_3(L)$ acts on the projective plane $P_2(L)$ of 1-dimensional subspaces of L^3; the projective plane $P_2(K)$ of K^3 then naturally embeds into $P_2(L)$ (henceforth we identify $P_2(K)$ with its image in $P_2(L)$). By restricting matrix entries with respect to the given basis, we obtain a subgroup $\mathrm{PGL}_3(K) \leqslant \mathrm{PGL}_3(L)$ stabilizing $P_2(K)$ inside $P_2(L)$. Given a 2-dimensional L-subspace M of L, then $\dim_K(M) = 4$, so $\dim_K(M \cap K^3) \in \{1, 2\}$. The two cases for $\dim_K(M \cap K^3)$ can be seen in the projective plane as follows: letting l_M be the line of $P_2(L)$ given by M, if $\dim_K(M \cap K^3) = 2$ then l_M contains a line of $P_2(K)$, whereas if $\dim_K(M \cap K^3) = 1$ then l_M contains a unique point $p_K(l_M)$ of $P_2(K)$, and we say l_M is **tangent** to $P_2(K)$.

We now define the **quadratic-extended projective plane** $\Pi =: P_2^L(K)$ to be the set of lines of $P_2(L)$ tangent to $P_2(K)$, and let $\mathrm{PGL}_3(K)$ act on Π in the natural manner. Consider a pair $\pi, \pi' \in \Pi$ with $\pi \neq \pi'$. Then π and π' intersect in a unique point p of $P_2(L)$. There are two possibilities: either $p \in P_2(K)$, in which case $p = p_K(\pi) = p_K(\pi')$ and we say π' is close to π, or $p \notin P_2(K)$, in which case we say π' is distant from π. Closeness is then a $\mathrm{PGL}_3(K)$-invariant equivalence relation $\sim_K$ on Π.

Claim 1: $\mathrm{PGL}_3(K)$ *is sharply transitive on triples* (p_0, p_1, p_2) *with the following properties:*

(i) $p_1, p_2 \in P_2(K)$*;*

(ii) $p_0 \in P_2(L) \backslash P_2(K)$*;*

(iii) $\{p_0, p_1, p_2\}$ *is not collinear in* $P_2(L)$*;*

(iv) The line through p_0 *and* p_i *is tangent to* $P_2(K)$ *for* $i = \{1, 2\}$*.*

It is clear that $\mathrm{PGL}_3(K)$ respects all the properties listed, so it acts on the specified set of triples. Clearly also $\mathrm{PGL}_3(K)$ is transitive on pairs of distinct points in $P_2(K)$, so it is enough to consider triples (p_0, p_1, p_2) where p_1 and p_2 are given: let us take $p_1 = Lv_1$ and $p_2 = Lv_2$, and let H be the subgroup of $\mathrm{GL}_3(K)$ stabilizing p_1 and p_2; note that H contains the diagonal matrices of $\mathrm{GL}_3(K)$. A point $p_0 \in P_2(L) \backslash P_2(K)$ satisfying conditions (ii), (iii) and (iv) takes the form Lw for $w = v_0 + \beta_1 v_1 + \beta_2 v_2$, such that $L = K(\beta_1) = K(\beta_2)$. Since $L = K \oplus K\alpha$, we see that there is an element $h \in H$ sending v_0 to $v_0 + a_1 v_1 + a_2 v_2$ for some $a_1, a_2 \in K$ and fixing v_1 and v_2, chosen so that $hw = v_0 + b_1 \alpha v_1 + b_2 \alpha v_2$ for some $b_1, b_2 \in K^*$. Then by applying a diagonal matrix, we can send hw to the specific vector $x = v_0 + \alpha v_1 + \alpha v_2$. Since $p_0 = Lw$ was given in general form, we deduce that $\mathrm{PGL}_3(K)$ acts transitively on the triples given in the claim. Consider now an element g of $\mathrm{GL}_3(K)$ stabilizing Lx, Lv_1 and Lv_2. Then

$$\exists c_1, c_2, d_0, d_1, d_2 \in K : gv_0 = d_0 v_0 + c_1 v_1 + c_2 v_2;\ gv_1 = d_1 v_1;\ gv_2 = d_2 v_2,$$

and hence

$$gx = d_0 v_0 + (c_1 + d_1 \alpha) v_1 + (c_2 + d_2 \alpha) v_2.$$

Since $gx \in Lx$ we have

$$d_0 = c_1 \alpha^{-1} + d_1 = c_2 \alpha^{-1} + d_2,$$

from which we see that $0 = c_1 = c_2$ and $d_0 = d_1 = d_2$, so g represents the trivial element of $\mathrm{PGL}_3(K)$. Thus $\mathrm{PGL}_3(K)$ acts sharply transitively on the given triples, proving the claim.

Claim 2: The stabilizer $\Lambda = \mathrm{PGL}_3(K)(\pi)$ *of a single tangent line* π *is of the form* $\mathrm{AGL}_1(L)$*.*

Take for instance the tangent line π spanned by Lv_0 and Lw, where $w = v_1 + \alpha v_2$ and consider $g \in \mathrm{GL}_3(K)$ preserving π: say g has matrix entries a_{ij} with respect to the standard basis. Then g stabilizes Lv_0 and

$$gv_0 = a_{00} v_0 + a_{10} v_1 + a_{20} v_2;\ gw = (a_{01} + a_{02}\alpha) v_0 + (a_{11} + a_{12}\alpha) v_1 + (a_{21}\alpha^{-1} + a_{22}) \alpha v_2,$$

so $a_{11} + a_{12}\alpha = a_{21}\alpha^{-1} + a_{22} \neq 0$ and hence (a_{21}, a_{22}) is determined by (a_{11}, a_{12}), and also $a_{10} = a_{20} = 0$. With respect to the ordered basis (v_0, w, v_2), then g has matrix entries b_{ij} such that $b_{10} = b_{20} = b_{21} = 0$ and $b_{ij} = a_{ij}$ for $ij \in \{00, 02, 12\}$; the remaining entries are

$$b_{01} = a_{01} + a_{02}\alpha; \quad b_{11} = a_{11} + a_{12}\alpha \neq 0; \quad b_{22} = a_{22} - a_{12}\alpha.$$

As an element of $\mathrm{GL}_3(K)(\pi)$, we see that g is determined by the values a_{00}, b_{01}, b_{11}, but we can choose independently $a_{00} \in K^*, b_{01} \in L, b_{11} \in L^*$. Setting $b_{11} = 1$ yields a normal subgroup F of $\mathrm{GL}_3(K)(\pi)$ consisting of those elements that act trivially on L^3/Lv_0, whereas setting $a_{00} = 1$ and $b_{01} = 0$ yields a subgroup F^* consisting of those elements that fix v_0 and stabilize Lw; we also have the centre of $\mathrm{GL}_3(K)$ inside $\mathrm{GL}_3(K)(\pi)$, which arises as those elements of $\mathrm{GL}_3(K)(\pi)$ with $b_{01} = 0$ and $b_{11} = 1$. Given the degrees of freedom of the parameters, we see that $\mathrm{GL}_3(K)(\pi) = (Z \times F) \rtimes F^*$. We see that the group $F \rtimes F^*$ acts faithfully on $Lv_0 + Lw$ as a copy of $\mathrm{AGL}_1(L)$, and hence

$$\Lambda = \mathrm{GL}_3(K)(\pi)/Z \cong \mathrm{AGL}_1(L).$$

completing the proof of Claim 2.

From Claim 1, we deduce that $\mathrm{PGL}_3(K)$ is sharply transitive on distant pairs in Π: distant pairs (π, π') are in bijection with triples as in Claim 1 by sending (π, π') to the triple $(p_K(\pi), p_K(\pi'), \pi \cap \pi')$. Thus we have a sharply 2-by-block transitive action of $\mathrm{PGL}_3(K)$ on Π with respect to the equivalence relation $\sim_K$, with block stabilizer $\mathrm{AGL}_2(K)$ and point stabilizer $\mathrm{AGL}_1(L)$.

The action of $\mathrm{PGL}_3(K)$ can be extended by any group R of field automorphisms of L that normalize K. Let R act on L^3 by acting on the coefficients of the standard basis vectors; this yields a natural semidirect product $\mathrm{PGL}_3(L) \rtimes R$. Then R preserves K^3, so there is a subgroup of the form $\mathrm{PGL}_3(K) \rtimes R$; in the induced action on $P_2(L)$, the subplane $P_2(K)$ is preserved by R, as is the set of tangent lines to $P_2(K)$. Thus R acts on Π and respects the block structure, yielding a 2-by-block-transitive (but not necessarily block-faithful) action of $\mathrm{PGL}_3(K) \rtimes R$ on Π. Since $\mathrm{PGL}_3(K)$ is already transitive on Π, the point stabilizers are likewise extended by a copy of R.

More variations on the space Π are possible; one that is relevant to the finite groups setting is the following. We suppose that L/K is Galois, $\mathrm{Gal}(L/K) = \langle \theta \rangle$; note that in this case, if $R \geqslant \langle \theta \rangle$ is a group of field automorphisms of L normalizing K, then θ is central in R. There is then a natural action of $\langle \theta \rangle$ on L^3 by acting on the coordinates in the standard basis, and hence an action of $\langle \theta \rangle$ on $P_2(L)$; the latter action fixes $P_2(K)$ pointwise and preserves the set of tangent lines, so we obtain an action of R on Π. For each $\pi \in \Pi$ there is thus a conjugate $\overline{\pi} := \theta(\pi)$ that is close to π, with $\overline{\overline{\pi}} = \pi$. Since $\mathrm{PGL}_3(K)$ acts by elements of the fixed field, it commutes with the action of θ; we also have R commuting with θ. Hence the pairs of conjugate tangent lines form a system of imprimitivity for the action of $\mathrm{PGL}_3(K) \rtimes R$, and we obtain another 2-by-block-transitive $(\mathrm{PGL}_3(K) \rtimes R)$-set Ω/θ. Indeed, since the action of θ has been factored out, we have an action of $\mathrm{PGL}_3(K) \rtimes (R/\theta)$ on Π/θ.

This means, for example, if e is a natural number and $|K| = p^e$, we have an action of $\mathrm{PGL}_3(K) \rtimes \langle \phi \rangle$ on Π, where ϕ is the Frobenius map on L, and hence has order $2e$, and the action of $\phi^e = \theta$ on Π is nontrivial but stabilizes each block, so the action is not block-faithful. The permutation group induced on the blocks in $\mathrm{P\Gamma L}_3(K) \cong \mathrm{PGL}_3(K)\langle \phi \rangle / \langle \theta \rangle$, and the action of the latter extends to a 2-by-block-transitive action on Π/θ with point stabilizer $\mathrm{A\Gamma L}_1(L)$.

In the rest of this subsection and the next, we will consider the remaining possibilities for G to have 2-by-block-transitive action on G/L, where G satisfies Hypothesis 3.9. The PD case and the exceptional case (c) of Proposition 3.12 have already been dealt with; we are thus left with case (b) of Proposition 3.12, where G/Z has socle $\mathrm{PSL}_3(q)$ and we have an action of L on $(V/\alpha_0)\backslash\{0\}$ as a transitive subgroup A of $\Gamma\mathrm{L}_2(q)$ that does not contain $\mathrm{SL}_2(q)$. We may thus assume G satisfies Hypothesis 3.20.

We recall from Lemma 3.11 the structure of $\overline{G}(\alpha_0, \alpha_1)$, and the subgroups $W_0, W_1 = sW_1s$ and $Z^* = W_0 \times W_1 \times Z$, which is a subgroup of $G_{\mathrm{GL}}(\alpha_0, \alpha_1)$. In the present context, we can write $\overline{G}(\alpha_0,\alpha_1)$ as $Z^*\overline{B}$, where

$$\overline{B} \cap \mathrm{GL}_{n+1}(q) =: \overline{B}_{\mathrm{GL}} = (s\langle x_0\rangle s \times \langle x_0\rangle);\ \overline{B} = \overline{B}_{\mathrm{GL}}\langle y_0\rangle.$$

Since $Z^* \leqslant G$, we obtain a similar description of $G(\alpha_0,\alpha_1)$ as Z^*B, where $B := G \cap \overline{B}$ is a subgroup of index t_G in $\overline{B}$, and likewise $B_{\mathrm{GL}} := G_{\mathrm{GL}} \cap \overline{B}$ has index t_G in $\overline{B}_{\mathrm{GL}}$. More precisely, recalling that G has elements $x = x_0^{t_G}$ and $y = x_0^{r_G}y_0$ for $0 \leqslant r_G < t_G$, we see that $B = B_{\mathrm{GL}}\langle y\rangle$ and B_{GL} consists of elements of the form $sx_0^a s x_0^b$ such that $a+b$ is a multiple of t_G.

Note that $G_{\mathrm{GL}}(\alpha_0,\alpha_1) \cap \langle y\rangle = \langle y^{e_G}\rangle$ and $|\langle y^{e_G}\rangle| = 2$. If $p=2$ then $y^{e_G} \in Z^*$, since Z^* is normal and of odd index in $G_{\mathrm{GL}}(\alpha_0,\alpha_1)$.

The main outstanding case we need to deal with is that of a QP action (recall Definition 3.10), where A is contained in a copy of $\Gamma\mathrm{L}_1(q^2) \leqslant \Gamma\mathrm{L}_2(q)$. As we saw in Example 3.26, there is indeed a 2-by-block-transitive action of $\mathrm{P}\Gamma\mathrm{L}_3(q)$ on a set Ω with point stabilizer $\mathrm{A}\Gamma\mathrm{L}_1(q^2)$, where Ω takes the form of the set of $\mathrm{Gal}(q^2/q)$-orbits on the quadratic-extended projective plane. In the next proposition we will learn that all QP actions are minor variations on this theme.

First consider L as a subgroup of $\overline{G}(\alpha_0)$. Up to conjugation, L will be contained in a group of the form

$$\overline{L}^{\Gamma\mathrm{L}_1} = WZ \rtimes \langle \hat{x}_0, y_0\rangle,$$

where $\langle \hat{x}_0, y_0\rangle$ corresponds to the subgroup of $\Gamma\mathrm{L}_1(q^2)$ of index e/e_G that contains $\mathrm{GL}_1(q^2)$.

Taking into account that $L \leqslant G$, in fact L is conjugate in $\overline{G}(\alpha_0)$ to a subgroup of

$$L^{\Gamma\mathrm{L}_1} = WZ \rtimes \langle \hat{x}, y\rangle,$$

which has index t_G in $\overline{L}^{\Gamma\mathrm{L}_1}$. Now $\overline{G}(\alpha_0) = G(\alpha_0)\langle \hat{x}_0\rangle$, and for $t \in \mathbb{Z}$ we see that

$$G(\alpha_0) \cap \hat{x}_0^t L^{\Gamma\mathrm{L}_1}\hat{x}_0^{-t} \leqslant L^{\Gamma\mathrm{L}_1}.$$

So in fact L is $G(\alpha_0)$-conjugate to a subgroup of $L^{\Gamma\mathrm{L}_1}$. So to understand QP actions of G, it is enough to consider G/L such that $L \leqslant L^{\Gamma\mathrm{L}_1}$.

For future reference, we name some subgroups of $L^{\Gamma\mathrm{L}_1}$ of small index: for $d_x, d_y \in \{1,2\}$, write

$$L_{d_x,d_y} = WZ \rtimes \langle \hat{x}^{d_x}, \hat{x}^{d_x-1}y^{d_y}\rangle;\ L^*_{21} = WZ \rtimes \langle \hat{x}^2, y\rangle.$$

Proposition 3.27. *Assume Hypothesis 3.9 with $n=2$ and Hypothesis 3.20. Let $L^{\Gamma\mathrm{L}_1}$ be as above, and write $\mathcal{H}$ for the class of subgroups L of $L^{\Gamma\mathrm{L}_1}$ such that $Z \leqslant L$ and such that G has 2-by-block-transitive action on G/L. Then the groups $L_{11}, L_{12}, L_{21}, L^*_{21}$ as above are all in distinct G-conjugacy classes, except if $p=2$, in which case $L_{11} = L_{21} = L^*_{21}$. The set $\mathcal{H}$ consists of those groups of the form L_{d_x,d_y}, where $d_x, d_y \in \{1,2\}$ and $d_xd_y = |L^{\Gamma\mathrm{L}_1} : L| \leqslant 2$, such that the following additional conditions are satisfied:*

(i) if e_G is even, then $d_x = d_y = 1$;

(ii) if $t_G = 3$, then $r_G \neq 0$, e_G is a multiple of 3 and $p^{e/e_G} \equiv 1 \mod 3$.

In particular, we have $0 \leqslant |\mathcal{H}| \leqslant 3$.

Moreover, if $L \in \mathcal{H}$, then

$$|L \cap sLs : Z| = \frac{4e_G}{|L^{\Gamma\mathrm{L}_1} : L|^2 t_G};$$

the action of G/Z on G/L is sharply 2-by-block-transitive if and only if $e_G = t_G$ (equivalently, $|G| = |\mathrm{GL}_3(q)|$) and either $d_y = 2$, or $p > d_x = 2$.

Proof. By Proposition 3.12, we only need to consider subgroups L of $L^{\Gamma L_1}$ such that $WZ \leqslant L$. Thus we may assume that $L = WZ \rtimes (L \cap \langle \hat{x}, y\rangle)$. Let $H_L = L(\alpha_1) \cap \langle \hat{x}, y\rangle$ and note that $H_L \leqslant B$.

Let $w \in W$, $z \in Z$ and $k, l \in \mathbb{Z}$, and let $g = wz\hat{x}^k y^l$. Then y^l stabilizes α_1 and z acts trivially on $P_2(q)$, so $g\alpha_1 = \alpha_1$ if and only if $w\hat{x}^k\alpha_1 = \alpha_1$. Since $w\hat{x}^k \in \mathrm{GL}_3(q)$ it is enough to consider the image of v_1. Our choice of $\hat{x}$ ensures that $\hat{x}^k v_1 = \nu_1 v_1 + \nu_2 v_2$ for some $\nu_1, \nu_2 \in \mathbb{F}_q$, and then $w\hat{x}^k v_1 = \nu_1 v_1 + \nu_2 v_2 + \nu_0 v_0$ for some $\nu_0 \in \mathbb{F}_q$. Thus if $w\hat{x}^k\alpha_1 = \alpha_1$, then $\nu_2 = 0 = \nu_0$ and $\nu_1 \in \mathbb{F}_q^*$, in other words $\hat{x}^k$ stabilizes α_1, and then also w stabilizes α_1, so $w \in W_0$. Thus for all L such that $WZ \leqslant L \leqslant L^{\Gamma L_1}$, we have

$$L(\alpha_1) = W_0 Z \rtimes H_L,$$

and hence

$$sL(\alpha_1)sL(\alpha_1) = W_1 Z s H_L s W_0 Z H_L \subseteq Z^* s H_L s H_L.$$

Write $G^+ = G(\alpha_0, \alpha_1)$ and $G^+_{\mathrm{GL}} = G_{\mathrm{GL}}(\alpha_0, \alpha_1)$. Under our current assumptions we see that the equation (5) from earlier is equivalent to

$$G^+ = Z^* s H_L s H_L. \tag{14}$$

Note that Z^*/Z is a p-Sylow subgroup of G^+_{GL}/Z. The intersection $\langle x, y\rangle \cap Z^*$ is therefore $\langle y^{e_G}\rangle$ in the case $p = 2$, and trivial otherwise.

Let $WZ \leqslant L \leqslant L^{\Gamma L_1}$ and write $d = |L^{\Gamma L_1} : L|$. Let $\mathcal{H}$ be the set of subgroups L of $L^{\Gamma L_1}$ such that G has 2-by-block-transitive action on G/L. We will proceed via a series of claims.

We have a restriction on d using the size of the large double coset of L in G.

Claim 1: If $L \in \mathcal{H}$, then

$$|L \cap sLs : Z| = \frac{4e_G}{d^2 t_G}.$$

In particular, $4e_G/(d^2 t_G)$ is an integer.

We have

$$|L : Z| = \frac{|L^{\Gamma L_1} : Z|}{d} = \frac{2q^2(q^2-1)e_G}{dt_G};$$

by comparison,

$$|G(\alpha_0) : Z| = \frac{q^3(q-1)(q^2-1)e_G}{t_G},$$

so

$$\begin{aligned}|G : Z| - |G(\alpha_0) : Z| &= |G(\alpha_0) : Z|(|G : G(\alpha_0)| - 1)\\ &= q^3(q-1)(q^2-1)(q^2+q)e_G/t_G = q^4(q^2-1)^2 e_G/t_G.\end{aligned}$$

Applying Corollary 2.2 to G/Z, we deduce that

$$|L \cap sLs : Z| = \frac{|L : Z|^2}{|G : Z| - |G(\alpha_0) : Z|} = \frac{4q^4(q^2-1)^2 e_G^2 t_G}{d^2 t_G^2 q^4 (q^2-1)^2 e_G} = \frac{4e_G}{d^2 t_G}$$

as claimed.

Since L acts transitively on lines of V other than α_0, we can calculate d as $d = |L^{\Gamma L_1}(\alpha_1) : L(\alpha_1)|$; in turn, given the form taken by $L(\alpha_1)$, we see that $|L^{\Gamma L_1}(\alpha_1) : L(\alpha_1)| = |H_{L^{\Gamma L_1}} : H_L|$, where $H_{L^{\Gamma L_1}} = \langle x, y\rangle$. The next claim puts more restrictions on the value of d.

Claim 2: Suppose $L \in \mathcal{H}$. If e_G is even, then $d = 1$; otherwise, $d \leqslant 2$.

We first consider integers $b > 0$ such that $x^a y^b \in H_L$ for some $a \in \mathbb{Z}$. We have $G^+ = G^+_{\mathrm{GL}}\langle y\rangle$ and $G^+_{\mathrm{GL}} \cap \langle y\rangle = \langle y^{e_G}\rangle$; since s centralizes the quotient G^+/G^+_{GL}, we see via (14) that $G^+ =$

$G^+_{\mathrm{GL}}H_L$. If e_G is even then we deduce that $x^a y \in H_L$ for some $a \in \mathbb{Z}$, and if e_G is odd then $x^a y^2 \in H_L$ for some $a \in \mathbb{Z}$.

Write $Q = H_L \cap B_{\mathrm{GL}}$ and $Q^{\Gamma\mathrm{L}_1} = H_{L^{\Gamma\mathrm{L}_1}} \cap B_{\mathrm{GL}}$. We have $Q^{\Gamma\mathrm{L}_1} = \langle x \rangle$ if $p = 2$ and $Q^{\Gamma\mathrm{L}_1} = \langle x, y^{e_G} \rangle \cong \langle x \rangle \times C_2$ if $p > 2$.

We claim next that Q contains x^l for l coprime to k, for all odd primes k. If $k = t_G = 3$ and $q \not\equiv 1 \mod 9$, or if k does not divide $q-1$, then $|\langle x \rangle|$ is coprime to k and the conclusion is clear. So assume that k divides $q-1$, and if $k = t_G = 3$, assume that $q \equiv 1 \mod 9$. Let R_k be group of k-th powers of B_{GL}. Then B_{GL}/R_k does not have a cyclic subgroup of index $\leqslant 2$, so after applying Lemma 3.16 to the group B/R_k, we see that Q is not contained in R_k. We deduce that Q contains x^l where l is coprime to k.

Since the previous paragraph applies to all odd primes k, we see that Q contains x^{2^a} for some $a \geqslant 0$; since Q also contains $x^b y$ or $x^b y^2$ for some $b \in \mathbb{Z}$, it follows that d is a power of 2. If $p = 2$ we deduce that in fact $x \in Q$, and the claim follows in this case.

We now assume $p > 2$ and consider powers of 2 dividing d. If $p > 2$ and e_G is odd, we see from Claim 1 that d cannot be a multiple of 4, so $d \leqslant 2$.

Now suppose $p > 2$ and e_G is even; in particular, q is an even power of p, so $q-1$ is a multiple of 4. We consider the quotient B/R_2 of B, where R_2 is the set of squares in B_{GL}; note that R_2 is normalized by s and contains y^{e_G}, so we can write

$$B/R_2 = B_{\mathrm{GL}}/R_2 \times \langle yR_2 \rangle;\ B_{\mathrm{GL}}/R_2 = \langle sx_0^3 sR_2 \rangle \times \langle x_0^3 R_2 \rangle.$$

In order to achieve (14) we see that $H_L R_2/R_2$ must contain a nontrivial element of B_{GL}/R_2 (since otherwise the intersection of $Z^*(sH_L sH_L)R_2/R_2$ with B_{GL}/R_2 would be contained in a cyclic subgroup). Thus H_L, and hence also Q, contains a nonsquare element of B_{GL}. Given the form of $Q^{\Gamma\mathrm{L}_1}$, this can only be achieved if L contains an odd power of x, and then by the previous paragraph we deduce that $x \in Q$. This completes the proof of the claim.

From now on, we can assume that $d \leqslant 2$ and that $L = WZ \rtimes \langle \hat{x}^{d_x}, \hat{x}^{\epsilon} y^{d_y} \rangle$, such that $d_x, d_y \in \{1,2\}$; $d_x d_y = d$; and $0 \leqslant \epsilon < d_x$. Note that if $p = 2$ then $\hat{x}$ has odd order, so $\langle \hat{x} \rangle = \langle \hat{x}^2 \rangle$; in that case we will set $d_x = 1$. Since L has index at most 2 in $L^{\Gamma\mathrm{L}_1}$, we have $y^2 \in L$. We also have $x \in L$: if $p = 2$ we know that $\hat{x} \in L$, while if $p > 2$ then $\hat{x}^2 \in L$ and x is an even power of $\hat{x}$. Thus $\langle x, y^2 \rangle \leqslant H_L$.

Claim 3: L acts transitively on $P_2(q)\backslash\{\alpha_0\}$ if and only if $\epsilon = d_x - 1$.

Suppose $\epsilon \neq d_x - 1$; then $d_x = 2$ and $\epsilon = 0$. Then $p > 2$, so $q+1$ is even, and since y stabilizes α_1, we see that the L-orbit of α_1 is the same as the $WZ\langle \hat{x}^2 \rangle$-orbit of α_1. The latter orbit has size $q(q+1)/2 < q(q+1)$, so L is not transitive on $P_2(q)\backslash\{\alpha_0\}$, which is incompatible with G having 2-by-block-transitive action on G/L.

On the other hand, suppose that $\epsilon = d_x - 1$; thus (d_x, ϵ) is either $(2,1)$ or $(1,0)$. In either case, we see that the L-orbit of α_1 contains the $W\langle \hat{x} \rangle$-orbit, which has size $q(q+1)$, so indeed L acts transitively on $P_2(q)\backslash\{\alpha_0\}$.

From now on we assume that $\epsilon = d_x - 1$, so $L = L_{d_x,d_y}$ where $(d_x, d_y) \in \{(1,1),(1,2),(2,1)\}$. We also assume $d_x = d_y = 1$ in the case that e_G is even. By Lemma 2.6 we have $L \in \mathcal{H}$ if and only if the equation (4) is satisfied; by Claim 3, L is transitive on $P_2(q)\backslash\{\alpha_0\}$, so (4) is equivalent to (5); and as noted earlier, (5) is equivalent to (14). In fact we can refine (14) further. Writing $Z^{**} := Z^*\langle sxs, x \rangle$ it is clear that (14) implies

$$G^+ = Z^{**} sH_L sH_L. \tag{15}$$

On the other hand, we see that $Z^*\langle x \rangle$ and $Z^*\langle sxs \rangle$ are normal in G^+. If $g \in G^+$ can be written as

$$g = z(sx^b s)x^a(sh_1 s)h_2$$

for $z \in Z^*$, $a, b \in \mathbb{Z}$ and $h_1, h_2 \in H_L$, then since $x \in H_L$, we can rearrange to express g as $z' s h_1' s h_2'$ for $z' \in Z^*$ and $h_1', h_2' \in H_L$. Thus (14) is equivalent to (15).

The next claim will determine which of the groups L_{d_x,d_y} belong to $\mathcal{H}$, with one more restriction appearing for $t_G = 3$. Write $f_G = e/e_G$.

Claim 4: We have $L \in \mathcal{H}$ if and only if one of the following holds:

(a) $t_G = 1$;

(b) $t_G = 3$, e_G is a multiple of 3, $r_G \neq 0$ and $p^{f_G} \equiv 1 \mod 3$.

If e_G is even then $y \in H_L$. If e_G is odd then $y^2 \in H_L$ and $y^{e_G} \in Z^{**}$, and we can write $y = y^{e_G+2a}$ for some $a \in \mathbb{Z}$. So in either case $y \in Z^{**}H_L$. If $t_G = 1$ then $G^+ = Z^{**}\langle y \rangle$, so $G^+ = Z^{**}H_L$ and the equation (14) is satisfied.

For the rest of the proof of the claim, we may assume $t_G = 3$; given Claim 1, we may then assume e_G is a multiple of 3. Let $x_* = s x_0 s x_0^{-1}$. The group $B^{**} = G^+/Z^{**}$ is then a metacyclic group $\langle x_* Z^{**}, y Z^{**} \rangle$ of order $3e_G$. The image of $L^{\Gamma \mathrm{L}_1}(\alpha_1)$, hence also H_L, in B^{**} contains yZ^{**} and has index 3; thus

$$Z^{**}L(\alpha_1) = Z^{**}\langle y \rangle.$$

Note that $s x_* s = x_*^{k+1}$ where $k = -2$. By Lemma 3.18, to achieve $L \in \mathcal{H}$ we must have $y x_* y^{-1} \in Z^{**} x_*^a$ where $a \equiv 1 \mod 3$, so in fact $y x_* y^{-1} \in Z^{**} x_*$. In particular, we may assume from now on that $p^{f_G} \equiv 1 \mod 3$.

We have $y = x_0^{r_G} y_0$ and s commutes with y_0 modulo Z^{**}; since

$$s x_0^{r_G} s = s x_0^{r_G} s x_0^{-r_G} x_0^{r_G} = x_*^{r_G} x_0^{r_G},$$

we see that $sys \in Z^{**} x_*^{r_G} y$. It is now easy to see that

$$Z^{**}\langle sys \rangle \langle y \rangle = Z^{**}\langle x_*, y \rangle$$

if and only if $r_G \neq 0$. In turn, $Z^{**}\langle x_*, y \rangle = G(\alpha_0, \alpha_1)$; by the discussion before the claim, we deduce that (15) is satisfied if and only if $r_G \neq 0$, completing the proof of the claim.

The next claim distinguishes the conjugacy classes of the groups in $\mathcal{H}$.

Claim 5: The groups of the form $L_{11}, L_{12}, L_{21}, L_{21}^*$ are not conjugate to one another in G, except for the case $L_{11} = L_{21} = L_{21}^*$, which happens when $p = 2$.

When $p = 2$ we can ignore L_{21} and L_{21}^* as duplicates of L_{11}. Since L_{11} properly contains the others, it is not conjugate to them; Claim 3 distinguishes L_{21}^* from the others. Thus we may assume $p > 2$ and we only need to show that L_{12} is not conjugate to L_{21}. It is enough to consider conjugacy in $G(\alpha_0)$; since WZ is normal in $G(\alpha_0)$ and contained in every $L \in \mathcal{H}$, we can consider the images $\tilde{L}$ of the groups $L \in \mathcal{H}$ in $G(\alpha_0)/WZ$. We argue that $|Z_{12}| > |Z_{21}|$, where Z_{d_x,d_y} is the centre of of $\tilde{L}_{d_x,d_y}$. The group $\tilde{L}_{d_x,d_y}$ takes the form

$$\tilde{L}_{d_x,d_y} = \langle \tilde{x}^{d_x t_G} \rangle \rtimes \langle \tilde{y}^{d_y} \rangle$$

such that $\tilde{x}$ has order $(q^2 - 1)$ and $\tilde{y}\tilde{x}\tilde{y}^{-1} = \tilde{x}^{p^{f_G}}$. (Here $\tilde{y}$ could be the image of y or of $\hat{x}y$ in $\tilde{L}_{d_x,d_y}$; the distinction is not important for the structure of $\tilde{L}_{d_x,d_y}$.) We see that

$$Z_{d_x,d_y} \cong C_{a_{d_x,d_y}} \times C_{b_{d_x,d_y}}$$

for some natural numbers a_{d_x,d_y} and b_{d_x,d_y}, where the first and second cyclic groups are generated by the powers of $\tilde{x}$ and $\tilde{y}$ respectively in Z_{d_x,d_y}. We can estimate the orders of the cyclic groups as follows:

$$a_{12} = \gcd((q^2-1)/t_G, p^{2f_G} - 1); \; a_{21} = \gcd((q^2-1)/2t_G, p^{f_G} - 1); \; 2b_{12}/b_{21} \in \mathbb{Z}.$$

In particular, $|Z_{12}|$ is a multiple of $|Z_{21}|(p^{f_G}+1)/2t_G$. So to have $|Z_{12}| \leqslant |Z_{21}|$, we need $p^{f_G}+1 \leqslant 2t_G$. Since p is odd, this only leaves the cases $p \in \{3,5\}, f_G = 1, t_G = 3$. However, these cases are incompatible with Claim 4. Thus $|Z_{12}| > |Z_{21}|$, so L_{12} and L_{21} are not conjugate in $G(\alpha_0)$, which proves the claim.

Putting the claims together, we have proved the characterization of $\mathcal{H}$ as stated in the proposition. It remains to characterize when the action is sharply 2-by-block-transitive. By Claim 1, for $L \in \mathcal{H}$, this occurs exactly when $4e_G = d^2 t_G$. Given that $t_G \in \{1,3\}$ and $d \in \{1,2\}$, the only solutions are $d = 2$ and $e_G = t_G \in \{1,3\}$. □

Writing $QP(d_x, d_y)$ for the 2-by-block-transitive G-set obtained in Proposition 3.27, we see that $QP(1,2) \cong P_2^{\mathbb{F}_{q^2}}(\mathbb{F}_q)$ and $QP(1,1) \cong P_2^{\mathbb{F}_{q^2}}(\mathbb{F}_q)/\theta$, where θ is the field automorphism $x \mapsto x^q$ of $\mathbb{F}_{q^2}$. It is not clear if the G-set $QP(2,1)$ has a natural geometric interpretation, or if its construction extends to groups defined over infinite fields.

3.7 Exceptional actions with socle $\mathrm{PSL}_3(q)$

To finish the classification of 2-by-block-transitive actions extending the action on a finite projective plane, Proposition 3.12 and Lemma 2.9 ensure that we have only finitely many groups to consider, which will give us the exceptional actions in the sense of Definition 3.10 (other than the action of $G = \mathrm{PSL}_5(2)$ on cosets of $C_2^4 \rtimes \mathrm{Alt}(7)$, which is also exceptional in the sense of Definition 3.10, but we dealt with it separately in Lemma 3.4). Specifically, we can assume G and L satisfy Hypotheses 3.9 and 3.20, and that there is a subgroup A of $\Gamma\mathrm{L}_2(q)$, acting transitively on nonzero vectors, such that neither $A \geqslant \mathrm{SL}_2(q)$ nor $A \leqslant \Gamma\mathrm{L}_1(q^2)$. Taking account of the possible field sizes q and groups between $\mathrm{PSL}_3(q)$ and $\mathrm{P\Gamma L}_3(q)$, there are a total of eleven groups G/Z to consider:

$$G/Z \in \{\mathrm{P\Gamma L}_3(q) \mid q \in 5,7,9,11,19,23,29,59\} \cup \{\mathrm{PSL}_3(7), \mathrm{PSL}_3(9), \mathrm{PSL}_3(19)\}. \tag{16}$$

In the next proposition we list all the 2-by-block-transitive actions of G/Z satisfying (16) that properly extend the action on $P_2(q)$ and are not PD or QP. We obtain a total of sixteen exceptional 2-by-block-transitive actions this way. All calculations of subgroups of small index and double coset enumerations are done using GAP.

Proposition 3.28. *Assume Hypotheses 3.9 and 3.20. Suppose that L neither contains $WZ \rtimes \mathrm{SL}_2(q)$, nor is contained in a copy of $WZ \rtimes \Gamma\mathrm{L}_1(q^2)$. Then G has 2-by-block-transitive action on G/L if and only if the action is as given in Table 1.*

Proof. By Proposition 3.12, we may assume that $L \geqslant W$ and that acts transitively on the nonzero vectors in V/α_0, so $L = WZ \rtimes A$ for some transitive subgroup A of $\Gamma\mathrm{L}_2(q)$ that is not contained in $\Gamma\mathrm{L}_1(q^2)$. The possibilities for A are then limited by Lemma 2.9, such that G/Z satisfies (16). As in the proof of Proposition 3.27, we have

$$|G(\alpha_0) : Z| = q^3(q-1)(q^2-1)e_G/t_G.$$

We will refer to the index of L in $G(\alpha_0)$ as the block size b, and write $a = q(q-1)/b \in \mathbb{Q}$. Then by Corollary 2.2, given $g \in G \backslash G(\alpha_0)$ we have

$$\begin{aligned} |L \cap gLg^{-1} : Z| &= \frac{|L/Z|^2}{|G(\alpha_0)/Z| q(q+1)} = \frac{|G(\alpha_0)/Z|}{b^2 q(q+1)} \\ &= \frac{q^3(q-1)(q^2-1)e_G}{b^2 q(q+1)t_G} = \frac{a^2 e_G}{t_G}. \end{aligned} \tag{17}$$

In particular, $a^2 e_G/t_G$ must be an integer. In the present situation, $e_G \in \{1,2\}$ and $t_G \in \{1,3\}$, so in fact a is an integer, in other words, b divides $q(q-1)$; indeed, b divides $q(q-1)/t_G$. If the action on G/L is 2-by-block-transitive, we see that it is sharply 2-by-block-transitive if and only if $e_G = t_G = a = 1$.

On the other hand, if $b \leqslant q$, then in most cases L contains $\mathrm{SL}_2(q)$ and hence gives rise to a PD action (assuming that G acts 2-by-block-transitively on G/L). The exception is if there is some proper subgroup $R(q)$ of $\mathrm{SL}_2(q)$ of index $\leqslant q$, which occurs only for $q = 5, 7, 9, 11$, with $|\mathrm{SL}_2(q) : R(q)| = 5, 7, 6, 11$ respectively, see [10, Satz 8.28]. In that case, we need to consider multiples of $|\mathrm{SL}_2(q) : R(q)|$ as possible values of b.

Write $\Gamma = \Gamma\mathrm{L}_3(q)$ and $S = Z\mathrm{SL}_3(q)$. We will take $G \in \{\Gamma, S\}$.

Write $\mathcal{L}_G$ for the set of proper groups L' of $G(\alpha_0)$ satisfying the following conditions:

(i) $L' \geqslant Z$;

(ii) $|G(\alpha_0) : L'|$ divides $q(q-1)/t_G$;

(iii) $|G(\alpha_0) : L'| > q$ or $|G(\alpha_0) : L'|$ is a multiple of $|\mathrm{SL}_2(q) : R(q)|$ (or both).

Calculations are performed on the quotient G/Z.

If $t_G = 1$, write $\mathcal{L}^*_G$ for the groups in $\mathcal{L}_G$ conjugate to a subgroup of $L^{\Gamma\mathrm{L}_1}$ as in Proposition 3.27. Then $\mathcal{L}^*_G$ contains four classes, one with block size $q(q-1)/2$ and the others with block size $q(q-1)$; these correspond to the four conjugacy classes of subgroups $L_{11}, L_{12}, L_{21}, L^*_{12}$ that appeared in Proposition 3.27. We can exclude these groups from consideration.

We split the rest of the proof according to the field size q.

$q = 5$. There are three conjugacy classes in $\mathcal{L}_G \backslash \mathcal{L}^*_G$, of block size $5, 10, 20$. These have representatives L of the form $L_5 \leqslant L \leqslant \mathrm{N}_G(L_5)$, where L_5 represents the unique class of subgroups of $G(\alpha_0)$ of the form $WZ \rtimes \mathrm{SL}_2(3)$; we note that $\mathrm{N}_G(L_5)$ takes the form $WZ \rtimes (\mathrm{SL}_2(3) \rtimes C_4)$. A double coset enumeration shows that G acts 2-by-block-transitively on G/L_5, hence also on G/L for any overgroup L of L_5 in $G(\alpha_0)$.

By (17) we have $|L \cap gLg^{-1}|/|Z| = a^2$. More precisely, we find that $(L \cap gLg^{-1})/Z \cong C_a^2$.

$q = 9$. Excluding subgroups of S, there are three conjugacy classes in $\mathcal{L}_\Gamma \backslash \mathcal{L}^*_\Gamma$: one of block size 12 and two of block size 24.

Double coset enumerations with respect to representatives of $\mathcal{L}_\Gamma \backslash \mathcal{L}^*_\Gamma$ now show that there is one exceptional 2-by-block-transitive action of Γ, of block size 12. The point stabilizer has the form

$$L'_9 = WZ \rtimes (\mathrm{SL}_2(5).D_8),$$

and the stabilizer of a distant pair takes the form $P'_9 = \mathrm{Sym}(3)^2 \times C_2$.

There are two conjugacy classes in $\mathcal{L}_S \backslash \mathcal{L}^*_S$, of block sizes 12 and 24. By double coset enumeration with respect to the remaining candidates, we obtain one exceptional 2-by-block-transitive action of S, of block size 12, which is obtained by restricting the 2-by-block-transitive action of Γ. We have

$$L'_9 \cap S = WZ \rtimes (\mathrm{SL}_2(5).C_4); \; P'_9 \cap S = \mathrm{Sym}(3)^2.$$

$q = 11$. There are four conjugacy classes in $\mathcal{L}_G \backslash \mathcal{L}^*_G$: one of block size 22, one of block size 55 and two of block size 110. The two classes of block size 110 have representatives of the forms

$$L_{11} = WZ \rtimes (\mathrm{SL}_2(3) \times C_5); \; L'_{11} = WZ \rtimes \mathrm{SL}_2(5).$$

We find by double coset enumeration that G has 2-by-block-transitive action on both G/L_{11} and G/L'_{11}; since the block size is exactly $q(q-1)$, the stabilizer of a distant pair is trivial in both cases. The remainder of $\mathcal{L}_G$ is now accounted for by

$$\mathrm{N}_G(L_{11}) = WZ \rtimes (\mathrm{GL}_2(3) \times C_5); \; \mathrm{N}_G(L'_{11}) = WZ \rtimes (\mathrm{SL}_2(5) \times C_5),$$

of block sizes 55 and 22 respectively; clearly G also has 2-by-block-transitive action on both $G/\mathrm{N}_G(L_{11})$ and $G/\mathrm{N}_G(L'_{11})$.

As in the case $q = 5$, we find in all cases that $(L \cap gLg^{-1})/Z \cong C_a^2$.

$q \in \{7, 19, 23, 29, 59\}$**.** There are five or six conjugacy classes in $\mathcal{L}_\Gamma$, of which four are accounted for by $\mathcal{L}_\Gamma^*$. The remaining classes consist of a class of block size $q(q-1)$, represented by a group $L''_q = WZ \rtimes A_q$, and the other class (if there is one) is represented by $\mathrm{N}_\Gamma(L''_q) = WZ \rtimes N_q$. The groups are as follows:

$$A_7 = \mathrm{SL}_2(3).2;\ N_7 = \mathrm{SL}_2(3).2 \times C_3$$
$$A_{19} = \mathrm{SL}_2(5) \times C_3;\ N_{19} = \mathrm{SL}_2(5) \times C_9$$
$$A_{23} = N_{23} = \mathrm{SL}_2(3).C_2 \times C_{11}$$
$$A_{29} = \mathrm{SL}_2(5) \times C_7;\ N_{29} = (\mathrm{SL}_2(5) \rtimes C_2) \times C_7$$
$$A_{59} = N_{59} = \mathrm{SL}_2(5) \times C_{29}.$$

Suppose $L = WZ \rtimes A$, where A is one of the groups listed above. The groups $L = L''_7$ and $L = L''_{19}$ are contained in a proper normal subgroup S of Γ, so in these cases, by Corollary 2.4, Γ does not have 2-by-block-transitive action on Γ/L. By contrast, for all the other candidates listed for L (including $WZ \rtimes N_7$ and $WZ \rtimes N_{19}$), we find by double coset enumeration that Γ has 2-by-block-transitive action on Γ/L.

The remaining groups are $S = Z\mathrm{SL}_3(q)$ for $q \in \{7, 19\}$. For each of them, there is only one conjugacy class in $\mathcal{L}_S$, represented by L''_q. By double coset enumeration, we find that S has 2-by-block-transitive action on S/L''_q for $q = 7$ but not for $q = 19$.

It remains to describe $(L \cap gLg^{-1})/Z$ for the possible 2-by-block-transitive actions with $q \in \{7, 19, 23, 29, 59\}$. In most cases, the answer is clear from (17), as $(L \cap gLg^{-1})/Z$ is trivial or has prime order. The exceptions are for $L = \mathrm{N}_\Gamma(L''_q)$ for $q \in \{7, 19\}$, in which case we find that $(L \cap gLg^{-1})/Z \cong C_3^2$, and for $L = \mathrm{N}_\Gamma(L''_{29})$, in which case we find that $(L \cap gLg^{-1})/Z \cong C_2^2$. □

Example 3.29. It will be useful for later applications to note the set of 2-by-block-transitive actions of G for $\mathrm{PSL}_3(q) \leqslant G \leqslant \mathrm{P\Gamma L}_3(q)$ and $2 \leqslant q \leqslant 5$ that properly extend the action on $P_2(q)$; these are listed in Table 2. For a QP action, the pair of numbers indicates the value of (d_x, d_y).

G	$G(\omega)$	type	$\|B\|$
$\mathrm{PSL}_3(2)$	$W \rtimes C_3$	QP(1,2)	2
$\mathrm{PSL}_3(3)$	$W \rtimes \mathrm{SL}_2(3)$	PD	2
	$W \rtimes \Gamma\mathrm{L}_1(9)$	QP(1,1)	3
	$W \rtimes C_8$	QP(1,2)	6
	$W \rtimes Q_8$	QP(2,1)	6
$\mathrm{PGL}_3(4)$	$W \rtimes (C_{15} \rtimes C_2)$	QP(1,1)	6
	$W \rtimes C_{15}$	QP(1,2)	12
$\mathrm{P\Gamma L}_3(4)$	$W \rtimes \Gamma\mathrm{L}_1(16)$	QP(1,1)	6
$\mathrm{PSL}_3(5)$	$W \rtimes \mathrm{SL}_2(5).C_2$	PD	2
	$W \rtimes \mathrm{SL}_2(5)$	PD	4
	$W \rtimes \Gamma\mathrm{L}_1(25)$	QP(1,1)	10
	$W \rtimes C_{24}$	QP(1,2)	20
	$W \rtimes (C_3 \rtimes C_8)$	QP(2,1)	20
	$W \rtimes (\mathrm{SL}_2(3) \rtimes C_4)$	exceptional	5
	$W \rtimes (\mathrm{SL}_2(3) \rtimes C_2)$	exceptional	10
	$W \rtimes \mathrm{SL}_2(3)$	exceptional	20

Table 2: 2-by-block-transitive actions with socle $\mathrm{PSL}_3(q)$ for $2 \leqslant q \leqslant 5$

Some remarks:

(1) There are no proper PD actions of $G = \mathrm{PSL}_3(2)$, and for a QP action, the parameters $(d_x, d_y) = (1, 1)$ do not give a proper extension of $P_2(q)$ either, because

$$\Gamma\mathrm{L}_1(4) = \Gamma\mathrm{L}_2(2) = \mathrm{SL}_2(2).$$

(2) For $q = 4$ there are no proper PD actions: this is because, in the notation of Proposition 3.14, we have $q - 1 = n + 1$, so M is the kernel of the projective determinant map on $G_{\mathrm{GL}}(\alpha_0)$. There are no QP actions of $G = \mathrm{PSL}_3(4)$ and $G = \mathrm{P\Sigma L}_3(4)$ because $t_G = 3$ and e_G is not a multiple of 3.

(3) In the QP action of $\mathrm{P\Gamma L}_3(4)$ or $\mathrm{PGL}_3(4)$ with block size 6, each block can be identified with a copy of $P_1(5)$, with $G([\omega])$ acting as $\mathrm{PGL}_2(5)$ (if $G = \mathrm{P\Gamma L}_3(4)$) or $\mathrm{PSL}_2(5)$ (if $G = \mathrm{PGL}_3(4)$) on the block. In the exceptional action of $\mathrm{PSL}_3(5) = \mathrm{P\Gamma L}_3(5)$ with block size 5, each block can be identified with a copy of $P_1(4)$, with $G([\omega])$ acting as $\mathrm{PGL}_2(4) \cong \mathrm{Sym}(5)$. These actions all appear as exceptional imprimitive rank 3 actions in [4]. The exceptional relationship between these 2-by-block-transitive actions of $\mathrm{P\Gamma L}_3(4)$ and $\mathrm{P\Gamma L}_3(5)$ plays a role in the structure of boundary-2-transitive actions on the $(21, 31)$-regular tree, see [14].

3.8 Proofs of main theorems

We can now prove the theorems from the introduction.

Proof of Theorem 1.1. Let $k \geqslant 3$ and suppose G is a finite block-faithful k-by-block-transitive permutation group, such that the blocks are not singletons. Then G acts k-transitively on the set X of blocks; let G_0 be the permutation group induced by G on X. By Corollary 2.4, G_0 is not of affine type. By Corollary 3.7, G_0 is not $\mathrm{Alt}(X)$ or $\mathrm{Sym}(X)$. By Proposition 3.24, G_0 is not an action of $\mathrm{PSL}_2(q) \leqslant G \leqslant \mathrm{P\Gamma L}_2(q)$ on the projective line. By Lemma 3.3, G_0 is not a Mathieu group. By the classification of finite 3-transitive permutation groups, all possibilities are eliminated and we have a contradiction. ☐

Corollary 1.2 now follows immediately from Theorem 1.1, together with the explanation given in the introduction of the relationship between general k-by-block-transitive actions and block-faithful k-by-block-transitive actions.

Proof of Theorem 1.3. We have a finite block-faithful 2-by-block-transitive permutation group G, acting on the set Ω and preserving some proper nontrivial system of imprimitivity; let $\omega \in \Omega$ and let $[\omega]$ be the block containing ω. Then G has faithful 2-transitive action on $G/G([\omega])$, and thus the pair $(G, G([\omega]))$ is subject to the classification of finite 2-transitive permutation groups; in particular, either this action is affine type or G is almost simple. By Lemma 2.3, $G([\omega])$ is the unique maximal subgroup of G containing $G(\omega)$. By Corollary 2.4, G has no nontrivial abelian normal subgroup, and hence G is almost simple. Let G_0 denote the permutation group given by the action of G on $\Omega_0 = G/G([\omega])$. We now go through the list of almost simple 2-transitive permutation groups.

By Corollary 3.7, G_0 cannot be a symmetric or alternating group in standard action. By Lemma 3.8, G_0 cannot belong to either of the families of 2-transitive actions of symplectic groups. By Lemma 3.2, we exclude the cases where $(G_0, |\Omega_0|)$ is in the set

$$\{(\mathrm{PSL}_2(11), 11), (\mathrm{Alt}(7), 15), (\mathrm{P\Gamma L}_2(8), 28), (\mathrm{HS}, 176), (\mathrm{Co}_3, 276)\}.$$

By Lemma 3.3, the only way G_0 can have socle a Mathieu group is if $G_0 = \mathrm{M}_{11}$ acting on 11 points; $G([\omega]) = \mathrm{M}_{10}$; and $G(\omega) = \mathrm{Alt}(6)$. This indeed produces a block-faithful 2-by-block-transitive action of M_{11}, which is line 1 in Table 1.

From now on we may assume the socle S of G_0 is a group of Lie type in a standard 2-transitive action. If S has Lie rank 1, the possibilities are all accounted for by Proposition 3.21, which also deals with case (c) of the theorem. Thus we may assume that the Lie rank is at least 2, which means that $\mathrm{PSL}_{n+1}(q) \leqslant G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$ and (up to isomorphism of permutation groups) we may assume $G([\omega])$ is a point stabilizer of the usual action of G on the projective n-space $P_n(q)$. Thus the action of G on Ω is PD, QP or exceptional.

PD case. All such actions are described by Proposition 3.14, which also accounts for case (a) of the theorem.

QP case. Here $n = 2$, and all such actions are described by Proposition 3.27, which also accounts for case (b) of the theorem.

Exceptional case. If $G = \mathrm{PSL}_5(2)$, there is one exceptional action described by Lemma 3.4, which is line 2 of Table 1, so let us assume $G \neq \mathrm{PSL}_5(2)$. Then by Proposition 3.12 we have $n = 2$, and L does not contain $\mathrm{SL}_n(q)$. We may thus assume Proposition 3.28 applies; the resulting actions are displayed on lines 3 to 18 of Table 1, completing the argument for case (d) of the theorem.

One can easily check that cases (a)–(d) are mutually exclusive as claimed; see Example 3.29 for more details on $\mathrm{PSL}_3(2)$ and $\mathrm{PSL}_3(3)$, which are to some extent special cases. □

Proof of Corollary 1.4. We suppose G acts on Ω, preserving an equivalence relation, such that G acts transitively on $\Omega^{[2]}$. Let $G([\omega]) \geqslant G(\omega)$ be the block and point stabilizers of the action, and let K be the socle of $G([\omega])$; since K is normal in $G([\omega])$ it is enough to show $K \leqslant G(\omega)$. If $G([\omega]) = G(\omega)$ the conclusion is clear, so assume $G([\omega]) > G(\omega)$. Then we are in the setting of Theorem 1.3.

If $\mathrm{PSL}_{n+1}(q) \leqslant G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$ for $n \geqslant 2$, we take $G([\omega])$ to be a point stabilizer of the usual action of G on the projective n-space $P_n(q)$, then $K = W$, and we have $K \leqslant G(\omega)$ by Proposition 3.12.

If the socle of G is of rank 1 simple Lie type, then we are in case (c) of Theorem 1.3 and $K \leqslant P$, where P is as in Proposition 3.21; again we have $K \leqslant G(\omega)$.

The only remaining case is the exceptional 2-by-block-transitive action of $G = \mathrm{M}_{11}$ on 22 points. In this case $G(\omega) = \mathrm{Alt}(6)$ is exactly the socle of $G([\omega]) = \mathrm{M}_{10}$. □

Proof of Theorem 1.5. We may assume that the action of G on Ω is sharply 2-by-block-transitive, but not sharply 2-transitive. This means the blocks are not singletons, and we are in the setting of Theorem 1.3.

If case (a) or (c) of Theorem 1.3 holds, then by Lemma 3.11 or Corollary 3.22 respectively, the action is not sharply 2-by-block-transitive.

If Theorem 1.3(b) holds, then the characterization of sharply 2-by-block-transitive action as in case (b) of the present theorem follows from Proposition 3.27.

If Theorem 1.3(d) holds, then inspecting Table 1, we are left with six sharply 2-by-block-transitive actions, with the field size q as specified. One can observe that all relevant values of q are primes not congruent to 1 modulo 3, so indeed $\mathrm{PSL}_3(q) = \mathrm{P\Gamma L}_3(q)$, and case (c) of the present theorem holds. □

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